Mathematical Reviews

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HISTORY, BIOGRAPHY

★**Clagett, Marshall. Greek science in antiquity.** Abelard-Schuman, Inc., New York, 1955. xii+217 pp.

Conte, Luigi. Il confronto dei poliedri regolari nella "Collezione Matematica" di Pappo. Period. Mat. (4) 32 (1954), 125–141, 261–278.

Kennedy, E. S. Parallax theory in Islamic astronomy. Isis 47 (1956), 33–53.

Sudō, Toshiichi. A study of the history of mathematics in Ryu-kyu. I, II. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 4 (1954), 165–177; 5 (1955), 67–82.

★**Matematika i mehanika v izdaniyah Akademiya Nauk SSSR. Bibliografiya. 2. 1936-1947. Sostavila V. P. Alekseeva pod redakciel akademika V. I. Smirnova. [Mathematics and mechanics in publications of the Academy of Sciences of the USSR. A bibliography. 2. 1936-1947. Compiled by V. P. Alekseeva under the editorship of academician V. I. Smirnov.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 515 pp. 17.35 rubles.

This volume contains a chronological list of all papers on mathematics and physics published by the Academy of Sciences during 1936–1947. In addition there is a bibliography of papers on the history of mathematics and mechanics, an author index and a subject index. Vol. 1 of this Bibliography listed the papers from 1728 to 1935 [Akad. Nauk SSSR, Moscow, 1936].

Szénássy, Barna. Die matematische Tätigkeit der Ungarischen Akademie der Wissenschaften in den Jahren 1830-1867. Acta Univ. Debrecen. 1 (1954), 5–28; addiatamentum ad 1 (1955), 3. (Hungarian. German summary)

Jarník, Vojtěch. Ten years of mathematics in liberated Czechoslovakia. Czechoslovak Math. J. 5(80) (1955), 291–307. (Russian)

★**Marczewski, Edward. Rozwój matematyki w Polsce. [The development of mathematics in Poland.]** Polska Akademia Umiejętności Kraków, 1948. 47 pp.

Infeld, Leopold. The history of relativity theory. Rend. Mat. e Appl. (5) 13 (1955), 270–281.

★**Gorelik, G. S. The work of academician A. A. Andronov on the theory of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 51–62 (1 plate). Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

Adams, C. R.; and Neugebauer, Otto. R. C. Archibald and mathematics libraries. Science 123 (1956), 622–623.

★**Assur, L. V. Issledovanie ploskih steržnevyh mehanizmov s nizšimi parami s točki zreniya ih struktury i klassifikacii. [Investigation of plane bar mechanisms with lower pairs from the viewpoint of their structure and classification.]** Edited by I. I. Artobolevskiĭ. Izdat. Akad. Nauk SSSR, 1952. 592 pp. (4 folded inserts, 1 plate). 23.20 rubles.

Assur's treatise was published originally in Izv. Sankt-Peterburgsk. Politehn. Inst. 20–24 (1912–1915). The volume also contains a review of the work by Žukoskiĭ, an essay by I. I. Artobolevskiĭ, L. V. Assur and his work on the theory of mechanisms, some notes on the work by Artobolevskiĭ, and a list of Assur's publications.

★**Bianchi, Luigi. Opere. Vol. IV. Parte I. Deformazioni delle quadriche, teoria delle trasformazioni delle superficie applicabili sulle quadriche.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1956. 481 pp. 4500 Lire.

For earlier volumes see MR 14, 524; 15, 276, 591; 16, 781. The present volume contains an introduction by R. Calapso discussing the papers.

★**Kol'man, È. Bernard Bol'cano. [Bernard Bolzano.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 224 pp. 9 rubles.

Hercigonja, Mira. Les pseudonymes de R. J. Bošković. Bull. Soc. Math. Phys. Serbie 7 (1955), 109–118. (Serbo-Croatian. French summary)

★**Roĭtenberg, Ya. N. The work of B. V. Bulgakov, corresponding member of the Academy of Sciences of the USSR, on the theory of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 68–75 (1 plate). Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

Natucci, A. In memoria di Alfredo Capelli (1855-1910). Period. Mat. (4) 33 (1955), 257–275.

★**Dinnik, A. N. Izbrannye trudy. Tom I. [Selected works. Vol. I.]** Izdat. Akad. Nauk Ukrainskoĭ SSR, Kiev, 1952. 151 pp. (1 plate). 11.50 rubles.

★**Dinnik, A. N. Izbrannye trudy. Tom II. Priloženie funkcil Besselya k zadačam teorii uprugosti. [Selected works. Vol. II. Application of Bessel functions to problems of the theory of elasticity.]** Izdat. Akad. Nauk Ukrainskoĭ SSR, Kiev, 1955. 223 pp. 14.65 rubles.

Vol. I contains a long monograph, Impact and com-

pression of elastic bodies [Izv. Kievsk. Politehn. Inst. 1909, kn. 4, 253–371] and a shorter paper on stability of bending. In addition there is a brief biography and a list of Dinnik's published works. Vol. II. contains the monograph of the subtitle [Izv. Donsk. Politehn. Inst. 2 (1913), part 2, 47–78; Izv. Ekater. Gorn. Inst. 1915, no. 2, 1–137].

Swift, J. D. Diophantus of Alexandria. Amer. Math. Monthly 63 (1956), 163–170.

Have

★ **Euclid. The thirteen books of Euclid's Elements translated from the text of Heiberg with introduction and commentary by Thomas L. Heath. 2nd ed. Vol. I: Introduction and Books I, II. Vol. II: Books III-IX. Vol. III: Books X-XIII and Appendix.** Dover Publications, Inc., New York, 1956. xi+432 pp.; i+436 pp.; i+546 pp. Paperbound: $1.95 per vol.; clothbound: $4.00 per vol.

Reprint by photo-offset of the Cambridge University Press edition of 1926.

★ **Euklidovi Elementi. Στοιχεια. Knj. 5, 6, 7. [Euclid's Elements. Στοιχεια. Bks. 5, 6, 7.]** Translated and annotated by Anton Bilimović. Srpska Akademija Nauka., Beograd, 1953, 1953, 1953, 1955, 1955; 48 pp., 31 pp., 58 pp., 57 pp., 58 pp.

For Books I–IV in this translation see MR **11**, 150; **12**, 311; **15**, 276.

Obituary: Nikolaĭ Mitrofanovič Krylov. Ukrain. Mat. Ž. **7** (1955), 347–359. (Russian)

A list of Krylov's published works is included.

Efimov, N. V. Nikolaĭ Ivanovič Lobačevskiĭ. (On the centenary of Lobačevskiĭ's death.). Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 3–15. (Russian)

Have

★ **Barbensi, Gustavo. Paolo Ruffini.** Accademia di Scienze Lettere e Arti, Modena, 1956. x+128 pp. (1 plate).

Have

★ **Schläfli, Ludwig. Gesammelte mathematische Abhandlungen. Band III.** Verlag Birkhäuser, Basel, 1956. 402 pp. 59.30 Swiss francs.

This third volume begins with an n-dimensional generalization of Chasles's Theorem on polar triangles: If two simplexes are related by a polarity, the joins of corresponding vertices are a set of $n+1$ associated lines, i.e., any $(n-2)$-space that meets n of them meets the remaining one too [cf. H. F. Baker, Proc. Cambridge Philos. Soc. **32** (1936), 507–520, p. 510; S. Beatty, Amer. Math. Monthly **51** (1944), 599]. In the next paper, written n 1866, Schläfli returns to the problem of finding the content of a spherical simplex, to which he had made important contributions fifteen years earlier [Ges. Math. Abh., vol. 1, 1950, pp. 227–281, vol. 2, 1953, pp. 219–270, Birkhäuser, Basel; MR **11**, 611; **14**, 833]. In an editorial postscript, J. J. Burckhardt points out that one of the expressions used here is essentially a Bessel function, providing a link between Schläfli's geometrical investigations and his eight papers on Bessel functions, which occupy nearly 150 pages of the present volume. These played an important role in the preparation of G. N. Watson's well-known "Treatise on the theory of Bessel functions" [Cambridge, 1944; MR **6**, 64]. Other papers deal with such varied topics as partial differential equations, the motion of a pendulum (with comments by P. Bernays), the general quintic equation, elliptic functions, orthogonal systems of surfaces, Riemannian geometry, the general cubic surface, multiply periodic functions, and the conformal mapping of a polygon on a halfplane.

The present volume has the same high quality of annotation and printing as the other two.

H. S. M. Coxeter (Toronto, Ont.).

Nevanlinna, Rolf. Erhard Schmidt, zu seinem 80. Geburtstag. Math. Nachr. **15** (1956), 3–6.

Lorent, H. Un géomètre belge méconnu. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 83-91.

The geometer in question is M. Thysbaerts (fl. 1750).

★ **Piven', V. D. The work of I. N. Vosnesenskiĭ, corresponding member of the Academy of Sciences of the USSR, in the field of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 63–67 (1 plate). Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

Dyson, Freeman J. Obituary: Hermann Weyl. Nature **177** (1956), 457–458.

Have

★ **Selecta Hermann Weyl.** Birkhäuser Verlag, Basel und Stuttgart, 1956. 592 pp. (1 plate). 48.90 francs suisses.

A selection of nineteen papers by Hermann Weyl published jointly by the Federal Institute of Technology in Zurich and the Institute for Advanced Study in Princeton on the occasion of his 70th birthday. The selection was made with the aid of the author himself. Misprints have been corrected and one short passage rewritten. In addition there is a complete list of Weyl's published works.

See also: Mal'cev, p. 823; Leech, p. 888.

FOUNDATIONS, MATHEMATICAL LOGIC

Schröter, Karl. Theorie des logischen Schliessens. I. Z. Math. Logik Grundlagen Math. **1** (1955), 37–86.

These are the first two chapters of a terse, but systematic exposition of the classical propositional calculus, the predicate calculus of first order with and without identity, and the monadic predicate calculus of second order. Natural deduction and the calculus of sequences are treated both syntactically and semantically. There are a few excursions into intuitionistic and many-valued logics. The present paper is mainly descriptive; the table of contents for the remaining three chapters suggests that they will contain a larger proportion of mathematical arguments.

G. Kreisel.

Sanchez-Mazas, Miguel. The theory of the syllogism developed in algebraic form. Theoria **2** (1954), no. 7–8, 95–109. (Spanish)

Quine, W. V. A way to simplify truth functions. Amer. Math. Monthly **62** (1955), 627–631.

In Fortsetzung seiner früheren Note [Amer. Math. Monthly **59** (1952), 521–531; MR **14**, 440] vereinfacht

Verf. das Verfahren, das zu jeder aussagenlogischen Formel eine kürzeste äquivalente Normalform liefert, noch weiter. *P. Lorenzen* (Bonn).

Christian, Curt. Die Elimination des Unendlichkeitstheorems in einem intensionalen Kalkül. Theoria **3** (1955), no. 9, 129–133.

The author criticizes certain developments in arithmetic in first-order systems because (i) the assumption of a finite universe leads to the existence of a greatest integer which, as is to be expected, has „paradoxical" properties, and (ii) the assumption of a finite universe is consistent. The author suggests a development of arithmetic in a modal (intensional) logic in which arithmetic propositions are represented by formulae of the form: it is necessary that. The (contingent) assumption of a finite universe evidently does not imply the necessity of a greatest integer or of the properties of this concept which the author criticizes. He gives no hint how to develop any but the most rudimentary parts of arithmetic, namely those propositions which are true modulo N, for each $N=1, 2, \cdots$. *G. Kreisel* (Princeton, N.J.).

★**Günther, Gotthard. Dreiwertige Logik und die Heisenbergsche Unbestimmtheitsrelation.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 53–59. Editions du Griffon, Neuchâtel, 1955.

★**Lorenzen, Paul. Über eine Erweiterung des finiten methodischen Rahmens.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 128–134. Editions du Griffon, Neuchâtel, 1955.

The author is concerned with a framework for the foundations of mathematics. He describes briefly the notions "finitist" and "intuitionist", as applied to definitions and proofs, and observes that the distinction between these concepts is not made precise in the literature with which he is familiar. He emphasizes that not only the interpretation of quantifiers, but also of the logical connectives is problematic. A non-finitist framework, in particular the use of non-constructive (non-recursive) definitions, is proposed because "the exclusion of such definitions would only be justified if one knew that all non-constructive concepts were problematic". His own restriction (to so-called definite concepts) excludes all impredicative definitions over a range of variables which has not been "fixed in advance". To justify it, he (i) assumes that the notion of integer is sufficiently clear to regard the class of integers as "fixed in advance" (though, from the point of view of formalized systems of arithmetic, which have non-standard models, μ-symbol definitions are impredicative), (ii) says that (some) impredicative definitions lead to set-theoretical paradoxes (according to his own standard, he should say why the simple theory of types is problematic since he excludes it, too). The notion of "definite" is obtained by iteration of "definite with respect to proof" (p-definite) and "definite with respect to refutation" (r-definite): $(Ex)A(x)$ is p-definite, i.e. for decidable A, x ranging over the integers, $(Ex)A(x)$ is provable if true, $(x)A(x)$ is r-definite. This is familiar from the theory of formally undecidable propositions where $(x)A(x)[(Ex)A(x)]$ is true [false] if undecidable. $(x)(Ey)A(x, y)$ is also definite on the author's definition, because, for each x, $(Ey)A(x, y)$ is p-definite, and the universal quantifier is r-definite, but $(x)(Ey)A(x, y)$ is neither p- nor r-definite: both true and false propositions of this structure can be formally undecidable. Apparently the concept of a definite (prenex) expression of arithmetic can be understood negatively as follows: let $(x)\mathfrak{A}(x)$ $[(Ex)\mathfrak{A}(x)]$ be (provably) "indefinite" if there is an n such that $\mathfrak{A}(n)$ is indefinite, and for all m either $\mathfrak{A}(m)$ holds [is refuted] or $\mathfrak{A}(m)$ is indefinite; then there can be no instance of an indefinite prenex formula of arithmetic [by induction, since $\mathfrak{A}(n)$ has less quantifiers than $(x)\mathfrak{A}(x)$]. The author's notion of a definite set seems closely related to Kleene's hyper-arithmetic hierarchy [Trans. Amer. Math. Soc. **79** (1955), 312–340; MR **17**, 4]: for, here quantification over ranges 'fixed in advance' is continued into the recursive second number class to yield hyper-arithmetic sets, in particular, hyper-arithmetic ordinals, but continuation into this class does not yield any new sets, by a result of Spector [J. Symb. Logic **20** (1955), 151–163, Th. 6; MR **17**, 570].

Reviewer's notes. A distinction between intuitionist and finitist mathematics is noted in Hilbert-Bernays (HB) [Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, e.g. pp. 344–347: undecided (quantified) premisses in implications are non-finitist (though intuitionist) because they express an assumption about an infinite totality, or, at least, refer to an hypothetical proof; this distinction is conceptually clear though, in practice, simple implications $(x)A(x)\rightarrow(x)B(x)$ (HB, p. 347) can be replaced by a finitist version $A[f(n)]\rightarrow A(n)$ with a suitable recursive f, which is quantifier-free. Contrary to the author's opinion, logical connectives in quantifier-free (decidable) formulae are not problematic because the truth-functional interpretation makes sense with decidable formulae. It should be noted that the author's restriction excludes free choice sequences, which are used both in intuitionist and finitist mathematics, e.g. HB, p. 167. *G. Kreisel.*

Łoś, J.; and Suszko, R. On the extending of models. II. Common extensions. Fund. Math. **42** (1955), 343–347.

[For part I see Łos, Fund. Math. **42** (1955), 38–54; MR **17**, 224.] The authors say that a quantifier-free disjunction $\beta_1\vee\beta_2\vee\cdots\vee\beta_n$ has alternating variables (a.v.) if no variable occurs both in β_i and β_j for $i\neq j$. Consider a first-order system whose theorems are the formulae of the class X, and let $\mathfrak{M}_t$ $(t\in T)$ be a family of models of X. Then there exists a model of X which is a common extension of all the $\mathfrak{M}_t$ if and only if the following condition is satisfied. For every quantifier-free disjunction $\beta_1\vee\beta_2\vee\cdots\vee\beta_n$ of X with a.v. and every set of n models $\mathfrak{M}_{t_i}$, $1\leq i\leq n$, $t_i\in T$, there exists at least one β_k, $1\leq k\leq n$, which is valid simultaneously in these n models. The proof uses A. Robinson's notion of a complete description [On the metamathematics of algebra, North-Holland Publ. Co., Amsterdam, 1951; MR **13**, 715]. *G. Kreisel.*

Mostowski, A. Examples of sets definable by means of two and three quantifiers. Fund. Math. **42** (1955), 259–270.

This is a continuation of the author's previous investigations [Fund. Math. **34** (1947), 81–112; Ann. Soc. Polon. Math. **21** (1948), 114–119; MR **9**, 129; **10**, 175]. A typical result is: let the set $Z_\gamma^{(5)}$ be defined to be E_y [$\lim_x 10^{-x}\gamma(x, y)$ exists and is integral] then the family $\{Z_\gamma^{(5)}\}_{\gamma\in\Phi}$ is identical with $\mathfrak{Q}_3^{(1)}$, where Φ is the set of primitive recursive functions. From this result follows

that if $U(n, x, y)$ is a (general recursive) function universal for the class of primitive recursive functions with two arguments, then the set of pairs (n, y) such that $\lim_x 10^{-x}U(n, x, y)$ exists and has an integral value belongs to the class $\mathfrak{Q}_3^{(2)}$ but not to $\mathbf{P}_3^{(2)}$. Thus the minimal number of quantifiers needed for defining this set is 3. *P. C. Gilmore* (University Park, Pa.).

Mostowski, A. Contributions to the theory of definable sets and functions. Fund. Math. **42** (1955), 271-275.

In this paper are some further results of investigations concerning sets and functions definable in elementary arithmetic which the author initiated in Fund. Math. **34** (1947), 81–112 [MR **9**, 129]. Typical results are: if $n\geqq 1$ then $\mathbf{P}_{n+1}^{(k1)}-\mathfrak{Q}_n^{(k1)}\neq 0\neq\mathfrak{Q}_n^{(k1)}-\mathbf{P}_n^{(k1)}$; calling a function f a selector for a set $X\subset R_{k+1}$ such that $\Pi_m\Sigma_m(m, m)\in X$ if $\Pi_m(m, f(m))\in X$, then recursively enumerable sets possess recursive selectors; if $g\in\mathfrak{Q}_1^{(k1)}$ and g is not recursive, the set $E_{(m,m)}$ $[m>g(m)]$ is in $\mathfrak{Q}_1^{(k+1)}$ and has no recursive selector; and finally, if $X, Y\in\mathfrak{Q}_n^{(k)}$ and $XY=0$, then there is a set $Z\in\mathbf{P}_n^{(k)}\cdot\mathfrak{Q}_n^{(k)}$ such that $X\subset Z$ and $YZ=0$, although that the theorem is not true for $X, Y\in\mathbf{P}_n^{(k)}$, if $n>0$, follows from a theorem by S. C. Kleene [Nederl. Akad. Wetensch. Proc. **53** (1950), 800–802; MR **12**, 71]. *P. C. Gilmore.*

Goodstein, R. L. On non-constructive theorems of analysis and the decision problem. Math. Scand. **3** (1955), 261–263 (1956).

This paper is concerned with the place of Rolle's theorem (RT) in the author's formulation of recursive analysis (r.a.) [Acta Math. **92** (1954), 171–190; MR **16**, 783]. What the author's work establishes, roughly speaking, is that even if one has verified the conditions of RT for functions $f(x)$ defined in r.a., e.g. with $f(0)=f(1)=0$, there is no recursive method of calculating an integer p such that $f'(x)$ has a zero in $(p^{-1}, 1-p^{-1})$. Let $q_e(n)$ be a (recursive) enumeration of primitive recursive functions of n; the rational functions $f_e(n, x)$ have the following properties: $f_e(n, 0)=f_e(n, 1)=0$; $f_e'(n, x)$ converges uniformly (in e and x) as $n\to\infty$; if $(n)[q_e(n)=0]$, then $f_e'(n, x)=0\Leftrightarrow x=n^{-1}$ and $\lim_n f_e'(n, x)=0$; if e_0 is the least integer for which $q_e(e_0)\neq 0$, then, for $n\geqq e_0$, $f_e(n, x)=f_e(e_0, x)$ and $f_e'(n, x)=0\Leftrightarrow x=e_0^{-1}$. The functions $\lim_{n\to\infty} f_e(n, x)$ satisfy RT uniformly, but there can be no recursive $p(e)$ such that, for some c, satisfying $p^{-1}(e)<c<1-p^{-1}(e)$, $\lim_{n\to\infty} f_e'(n, c)=0$, since this would mean that $(n)[q_e(n)=0]$ if $(n)[n\leqq p(e)\Rightarrow q_e(n)=0]$. The author states (bottom of p. 261) that his construction establishes the existence of a function (presumably of his class) which satisfies RT, but not the following condition (IRT): There exists a recursive sequence (of rationals) c_k, a recursive ν_k and an integer p such that, for each numeral n, $n\geqq\nu_k\Rightarrow|f'(n, c_k)|<10^{-k}$ and $p^{-1}<c_k<1-p^{-1}$ are both provable (for some unexplained reason the author wants these formulae to be proved in an "unspecified" formalization of recursive analysis with the free variable k and not only for each numeral k). This statement is false; the recursiveness of c_k and ν_k is irrelevant, and the only recursiveness condition needed, namely on p, is not stated: let $\nu_k\geqq 10^{(k+1)/4}$, and $c_k=\frac{1}{2}$ if $q_e(n)=0$ for $n\leqq\nu_k$, otherwise $c_k=e_0^{-1}$, which makes ν_k and c_k recursive functions of e and k; now consider any e: if $(n)[q_e(n)=0]$, take $p=3$, otherwise $p=e_0+1$, when IRT is satisfied. The author's mistake is due to an invalid application of the law of the excluded middle: while IRT, which has an arithmetization $(e)(Ec)(E\nu)(Ep)(n)(k)R(e, n, k, c, \nu, p)$, cannot be satisfied by a recursive function $p(e)$, this does not mean, classically or intuitionistically, that

$$(Ee)-(Ec)(E\nu)(Ep)(n)(k)R(e, n, k, c, \nu, p)$$

holds. *G. Kreisel.*

Myhill, John. Solution of a problem of Tarski. J. Symb. Logic **21** (1956), 49–51.

The problem is stated on p. 19 of UT [Undecidable theories, North-Holland Publ. Co., Amsterdam, 1953; MR **15**, 384]: does every essentially undecidable theory have an essentially undecidable, finitely axiomatizable subtheory? The author's counter-example is a theory which contains only a finite number of non-logical constants, namely 0, + (though this is not used), and S (successor) and two binary predicate symbols P and Q; the axioms state (i) that $P(S^n0, S^m0)[Q(S^n0, S^m0)]$ are equivalent over the numerals to the two primitive recursive predicates $W_0(n, m)[W_1(n, m)]$, constructed by Kleene on p. 308 of IM [Introduction to metamathematics, Van Nostrand, New York, 1952; MR **14**, 525],

(ii) $(x)(y)(z)[P(x, y)\to\ulcorner Q(x, z)]$

this axiom is consistent, because $(Ey)W_0(n, y)$, $(Ey)W_1(n, y)$ are disjoint. This theory (T) is essentially undecidable since $(Ey)W_0(n, y)$, $(Ey)W_1(n, y)$ are not recursively separable. The finitely axiomatizable subtheories of T are undecidable, but not essentially so because P and Q may then be interpreted as representing certain finite sets of pairs of integers. The author states that T provides a negative solution to another problem of UT: is T_1 necessarily undecidable if T_1 and T_2 are compatible theories with the same constants, and T_2 is essentially undecidable and axiomatizable? The answer is negative even for theories with a finite number of constants (personal communication, H. Putnam), but this does not follow from the author's construction because the decidable extensions of the subtheories of T mentioned above, are not compatible with T. In the reviewer's negative solution (for theories with infinitely many constants) on p. 385 of MR **15**, the following obviously necessary set of axioms was inadvertently omitted, line 4 before 'M': let γ be a recursive function of n and e such that $P_{\gamma(n,e)}(i)$ is equivalent to $T_1(n, e, i)$, where T_1 is the predicate in Kleene's normal form on p. 288 of IM; then, for each e and n, add the axiom

$$(x)(y)[A_{\gamma(n,e)}(x)\&x\neq y\cdot\to\neg A_{\gamma(n,e)}(y)].$$

On the other hand the revieuwer's solution [MR **15**. 385, 1st column, lines 7–9] of the problem stated at the beginning of his revieuw contains only a finite number of non-logical constants, contrary to the author's assertion on p. 51, l. 7.

G. Kreisel.

Henkin, Leon. Two concepts from the theory of models. J. Symb. Logic **21** (1956), 28–32.

A class K of algebraic structures has the finite persistence property (fpp) means: every sentence H of the lower predicate calculus (pc) has a finite model in K (K-model), provided (i) H has a K-model, and (ii) every structure of K which is an extension of a K-model of H, is itself a model of H. A class K has the finite embedding property (fep) if every finite set α of every structure in K can be isomorphically embedded in some finite structure of K. The author shows that fpp→fep, and, for classes K which consist of all models of a set of sentences of (pc), fep→fpp. Some very plausible applications to familiar algebraic concepts are made. *G. Kreisel.*

Robinson, Abraham. Note on a problem of L. Henkin. J. Symb. Logic **21** (1956), 33–35.

The problem considered occurs in the paper reviewed above: to find a necessary and sufficient syntactical condition for the persistence of a sentence H of (pc) with respect to a set K of sentences of pc; 'persistence' means: if an extension of a model which satisfies both H and K, is a model of K, then it is a model of H. The author shows that the following condition is satisfactory: there is a sentence Y (in prenex form) whose quantifiers are all existential, such that $H \leftrightarrow Y$ can be proved in pc from K.

G. Kreisel.

★**Robinson, Abraham. Complete theories.** North-Holland Publishing Co., Amsterdam, 1956. vii+129 pp. 13.50 guilders.

The purpose of this book is to define two concepts (model-completeness and bounding transform), study their properties, and apply them to the so-called elementary theories of certain mathematical structures. A skeletal outline of the development is given in the following chapter-by-chapter account.

Chapter I. General introduction. After a brief description of the first-order functional calculus, a mathematical structure is defined as a non-empty set of constants together with a non-empty set of relations. The diagram of a structure is the set of all statements $R^*(a_1, \cdots, a_n)$, where R is one of the given relations, each a_i is one of the given constants, and R^* is R or $\sim R$ according as $R(a_1, \cdots, a_n)$ is true for the structure or not. The chapter concludes with a discussion of the Gödel completeness theorem and of a criterion (Vaught's test) for the completeness of a set of statements. (A set K of statements is complete if for every statement X, involving only the constants and relations of K, either X or $\sim X$ is deducible from K, or, in other words, if every such X is decidable in K.)

Chapter II. Model-completeness. A non-empty consistent set K of statements is model-complete if, for every structure M that is a model of K and contains no relations other than those in K, every statement (involving only the constants and relations in K) is decidable in the union of K with the diagram of M. The purpose of the chapter is to examine this concept (and a generalized version called partial model-completeness). Several of the applications of the theory are based on the fact that the following condition (on a non-empty consistent set K of statements) is necessary and sufficient for model-completeness: if a primitive statement involving only the constants and relations of K is true for a model of K, then it is true for all extensions of that model. (A primitive formula is a conjunction of atomic formulas and their negations, preceded by a string of existential quantifiers.)

Chapter III. Model-complete groups and fields. It is proved that several well-known kinds of mathematical structures are model-complete. Typical result: the elementary theory of an algebraically closed field with a non-trivial valuation in an ordered abelian group is model-complete.

Chapter IV. Completeness. The connections between completeness and model-completeness are studied. The tool for obtaining most of the results is the fact that if a model-complete set of statements has a prime model, then it is complete. (A prime model of K is a model M of K such that every model of K contains a submodel isomorphic to M.) This fact, in conjunction with the preceding results on model-completeness, implies the completeness of several theories.

Chapter V. Applications. This chapter reformulates and extends the results of the preceding one. The results usually take the form of "transfer principles". One of the simplest examples of such a principle is the assertion that a true statement about complex numbers (formulated in the first-order functional calculus in terms of equality, addition, and multiplication) is true for every algebraically closed field of characteristic zero. Several other (and more complicated) examples from algebra and algebraic geometry are discussed.

Chapter VI. Syntactical transforms. A syntactical transform is a correspondence that associates with every element of a certain class C of statements (in the first-order functional calculus) another statement obtained from the given one by a definite formal rule. Examples of syntactical transforms are the passage from a statement to its Skolem normal form, and the well-known process of relativisation. The emphasis in this chapter is on a very special case, called the bounding transform. This transform (too complicated to describe here) is designed to apply chiefly to the theory of ordered structures (especially ordered abelian groups). A trivial example (which nevertheless illustrates the purpose of the theory) is the passage from the statement that the sum of x and y exists to the statement that the sum of x and y (exists and) is uniformly bounded in terms of given bounds on x and y. In the general theory it is shown that the provability of certain statements implies the deducibility of their bounding transforms. *P. R. Halmos.*

Skolem, Th. A critical remark on foundational research. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 100–105.

This is a plea for a more consistent approach to foundational research. The classical set theory is not a reliable instrument but it is nevertheless frequently used. For example, the concepts of universal validity and satisfiability for formulas of the predicate calculus are set-theoretic concepts. The author urges a formalistic interpretation of these concepts in terms of provability within a formal system. Difficulties in the interpretation of quantification theory with an infinite range of variation are also recognized. Again a strictly formalistic interpretation circumvents the difficulties. Also a constructivistic interpretation of quantification can be attempted although it is suggested that a platonistic conception of mathematics is involved which the author is forced to reject from a finitary standpoint. The author then suggests three possible ways of providing foundations for mathematics. (1) Develop formal theories within the predicate calculus interpreting the calculus formalistically. (2) Begin with primitive recursive arithmetic and introduce into it further recursive functions obtaining in a creative way ever more extensive theories. Quantifiers are only to be introduced as shorthand for statements which can be interpreted finitistically in terms of statements with free variables. (3) Translate mathematical statements concerning an infinity of numbers into statements concerning only a finite number. The author concludes with "Probably one can say that the possibility of finitization of a mathematical theory is a criterion of its soundness". *P. C. Gilmore* (University Park, Pa.).

Gandy, R. O. On the axiom of extensionality. I. J. Symb. Logic **21** (1956), 36–48.

Extensional equality in the simple theory of types of

Church [same J. 5 (1940), 56–68; MR 1, 321] is introduced by taking it as primitive for type ι, defining it to be logical equivalence for type 0, and defining it by induction for higher types:

E_{000} for $\lambda p_0 q_0 \cdot p \equiv q$; $E_{0(\alpha\beta)(\alpha\beta)}$ for $\lambda f_{\alpha\beta} g_{\alpha\beta} \cdot (x_\beta)(E_{0\alpha\alpha}(fx)(gx))$; $\mathbf{A}_\alpha \overset{E}{=} \mathbf{B}_\alpha$ for $E_{0\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha$.

The system C is that which has rules I–VI and axioms 1–4, 5^α and 6^α of the simple theory of types and the following axioms in place of the axiom of infinity:

N.1 $x_\iota \overset{E}{=} x_\iota$; N.2 $x_\iota \overset{E}{=} y_\iota \& z_\iota \overset{E}{=} y_\iota \cdot \supset x_\iota \overset{E}{=} z$;

N.3 $\sim 0_\iota \overset{E}{=} S x_\iota$; N.4 $x_\iota \overset{E}{=} y_\iota \equiv S x_\iota \overset{E}{=} S y_\iota$.

E^α, D^α, S.1$^\alpha$ and S.2$^\alpha$ are respectively,

$x_\alpha \overset{E}{=} y_\alpha \supset x_\alpha = y_\alpha$, $f_{0\alpha} x_\alpha \& (y_\alpha)(f_{0\alpha} y \supset y \overset{E}{=} x_\alpha) . \supset f_{0\alpha}(\iota f_{0\alpha})$,

$(\exists s_{\alpha(0\alpha)})(f_{0\alpha})((\exists x_\alpha)(fx) \supset f(sf))$, and

$(\exists s_{\alpha(0\alpha)})(f_{0\alpha})(g_{0\alpha})((\exists x_\alpha)(fx) \& f \overset{E}{=} g . \supset . f(sf) \& sf \overset{E}{=} sg)$.

In this paper are given syntactical proofs of the relative consistency of C with the axioms E^α to C, the relative consistency of C with the axioms D^α and E^α to C with the single axiom D^ι, and the relative consistency of C with the axioms E^α, S.1$^\alpha$ and S.2$^\alpha$ to C with the axioms S.2$^\alpha$.

P. C. Gilmore (University Park, Pa.).

Christian, Curt. A proof of the inconsistency of Quine's system, "Mathematical Logic (1951)". Theoria 3 (1955), no. 9, 135–136.

There is an error in line 8 of the second column on p. 135. It should read "$\dagger 191 \cdot {}'y \in \imath x \equiv . x \in V . y \in \imath x'$". This does not lead to an inconsistency. *I. Novak-Gál.*

Christian, Curt. Zwei Theoreme über die Einerklasse. Theoria 3 (1955), no. 9, 118.

The author proves two theorems on unit classes. It is not clear which system of set theory is used, but the results hold in most of the common set theories.

I. Novak-Gál (Ithaca, N.Y.).

Moch, François. On peut éviter les antinomies classiques sans restreindre la notion d'ensemble. C. R. Acad. Sci. Paris 242 (1956), 1402-1404.

In this note the author suggests that the formation of sets be left unrestricted as in the naive classical set theory and that the paradoxes of this theory be avoided by not only recognizing true and false sentences of this theory but a third category of sentences as well. *P. C. Gilmore.*

Moch, François. Des antinomies classiques à la Logique de Mme. Février-Destouches. C. R. Acad. Sci. Paris 242 (1956), 1562–1563.

Continuing the proposal of the paper reviewed above the author suggests that set theory should be developed within the three-valued logic of Mme. Février-Destouches.

P. C. Gilmore (University Park, Pa.).

★ **Isaye, G. Formalisme, intuitionnisme et métaphysique.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 121–127. Editions du Griffon, Neuchâtel, 1955.

★ **Stipanić, Ernest. Le principe de la permanence dans les mathématiques et la négation de la négation.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 138–143. Editions du Griffon, Neuchâtel, 1955.

★ **Baatard, François. Probabilités, systèmes logiques et systèmes cohérents.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 39–40. Editions du Griffon, Neuchâtel, 1955.

del Busto, Eduardo H. Probability and Carnap's inductive logic. Theoria 3 (1955), no. 9, 119–128. (Spanish)

An exposition is given of the principal works of R. Carnap on the foundations of probability [Philos. and Phenomenol. Res. 5 (1945), 513–532; 6 (1946), 590–602; Logical foundations of probability, Univ. of Chicago Press, 1950; The continuum of inductive methods, ibid., 1952; MR 7, 189; 8, 245; 12, 664; 14, 4]. *L. J. Savage.*

von Juhos, Béla. Deduktion, Induktion und Wahrscheinlichkeit. Methodos 6 (1954), 259–278.

García Bacca, Juan David. The notions of "fact" and "datum" in classical and modern physics. Theoria 3 (1955), no. 9, 17–24. (Spanish)

★ **Kustaanheimo, Paul. Über die Versuche, ein logisch finites Weltbild aufzubauen.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 60–66. Editions du Griffon, Neuchâtel, 1955.

ALGEBRA

on order

★ **Kochendörffer, Rudolf. Einführung in die Algebra.** Deutscher Verlag der Wissenschaften, Berlin, 1955. xii+316 pp. DM 23.60.

This book contains a fairly complete account of what might be called classical abstract algebra. In the first part, elementary properties are given of the natural number system, groups, rings, fields, and polynomial domains. A more complete discussion of fields is given in the second part. This includes field extensions, the Galois theory, and valuation theory. The third and final part contains a discussion of modules, finite-dimensional algebras, and elementary representation theory.

R. E. Johnson (Northampton, Mass.).

Have

★ **Ballieu, Robert, et Simonart, Fernand. Algèbre.** Librairie Universitaire, Louvain; Gauthier-Villars, Paris, 1955. xvi+356 pp.

This text is intended as a college algebra course. It is complete enough to include even some material which some graduate students could study with profit. The two first chapters are strongly influenced by the work of Bourbaki; in these chapters the authors introduce the concepts group, ring, field, matrix, vector space. The style of the remaining chapters is more nearly classical. Among other subjects, determinants, quadratic forms, and symmetric functions are mentioned; transformations of polynomial equations are also discussed. The last chapter (75 pp.) is on the solution of polynomial equations. There

are numerical examples as well as theorems on bounds for roots, separation of roots, and even Grace's theorem: At least one zero of the derivative of

$$(x^2-1)(x^{n-2}+ax^{n-3}+\cdots)$$

has modulus not exceeding cot (π/n). *J. L. Brenner.*

Fleischer, Isidore. A note on subdirect products. Acta Math. Acad. Sci. Hungar. **6** (1955), 463–465. (Russian summary)

Let θ_1 and θ_2 be congruence relations on an algebra A. If $\theta_1 \cap \theta_2$ is the identity relation, then A has a natural representation as a direct product of A/θ_1 and A/θ_2. The author shows that this representation consists of all pairs whose components have the same image by a pair of homomorphism if and only if θ_1 and θ_2 permute. This generalizes a result of Fuchs [same Acta 3 (1952), 103–120; MR **14**, 612]. *R. P. Dilworth* (Pasadena, Calif.).

Linear Algebra, Polynomials, Invariants

★**Kořínek, Vladimír. Základy algebry. [Elements of algebra.]** Nakladatelství Československé Akademie Věd, Praha, 1953. 488 pp. Kčs 320.

This is a text-book for mathematics students in their first and second years at the university. Therefore it is kept on a fairly elementary level. It is written, however, with high scientific aspirations, recognizing the fact that "of all branches of mathematics taught in the first year, algebra is the one most suitable to educate the reader for an exact logical development of his thought. Therefore much care has been given to a consistent logical structure of the content" (from the preface). Of the traditional material of an introductory algebra course not only the numerical methods for the solution of algebraic equations, but also the elements of group theory have been omitted. Nevertheless the book tries successfully to build up a sound knowledge of the fundamentals of classical algebra in modern terms, thus creating a basis for the study of higher analysis as well as for later specialization in higher algebra.

Here is a short survey of the contents. Part I. Elementary notions and laws of algebra. Chap. I (pp. 17–211) starts with the system of natural numbers, takes the induction principle for granted and proves from here (as may be mentioned by way of illustration) the transitivity of equality. After a primitive introduction of the negative numbers follows a deduction of the basic laws of arithmetic. The notion of integral domain is followed by the introduction of rational numbers, the notion of field and an exposition of the elementary theory of numbers. Chap. II (pp. 121–141). Real numbers (very short, referring for all details to van der Waerden) and complex numbers. Chap. III (pp. 142–208). Polynomials (including irreducibility) and rational functions. Part II. Linear Algebra. Chap. IV (pp. 209–247) introduces the n-dimensional vector space over an integral domain J as the set of all n-tuplets with coordinates in J, thus taking the point of view of the student who has had his usual course in coordinate geometry and is not prepared for a theory of abstract spaces. After a few statements about matrices (rank etc.) a theory of linear equations without determinants is developed. Chap. V (pp. 248–302) Determinants, up to Laplace's expansion, and their application in linear equations. Chap. VI (pp. 303–335) contains the elements of matrix algebra, linear substitutions, quadratic forms. — Part III. Algebraic equations in one unknown. Chap. VII (pp. 337–371) on roots presents, after carefully preparing the necessary analysis, an "elementary" proof of the so-called fundamental theorem of algebra (of which a second proof is given in the appendix of the book), followed up by the abstract construction of a root field for an algebraic equation over an arbitrary field. Chap. VIII (pp. 372–403). Symmetric functions (main theorem). Chap. IX (pp. 404–463) deals with $x^n=1$ and $x^n=a$ (formal properties only), equations of degree 2, 3, 4, reciprocal equations, and constructions with ruler and compass.

The book is written carefully and in a broad style. Each of the 53 sections is followed by a set of graded examples for the reader. Answers to the more difficult questions are provided at the end of the book. For further examples the author cites the "Collection of problems in higher algebra" by D. K. Faddeev and I. S. Sominskiĭ (in Russian) [Gostehizdat, Moscow-Leningrad, 1952; MR **14**, 1055]. *H. Schwerdtfeger* (Melbourne).

★**Gröbner, Wolfgang. Matrizenrechnung.** Verlag von R. Oldenbourg, München, **1956.** 249 pp. DM 23.00.

This is a text on elementary matric theory. Determinants are thoroughly discussed by means of Grassman algebra. All of the usual topics are presented. Furthermore, the text treats the spectral theory for finite matrices, elementary inequalities on the characteristic roots of matrices, approximate solution of linear equations, and automorphs of matrices. The exercises include many which sketch extensions of textual results and these are freely used to shorten the formal presentation. There are many geometric illustrations but the treatment does not emphasize the underlying vector spaces and their linear transformations to any great extent.

The author has packed a great deal of detailed information on matrices into relatively few pages. It is not surprising, then, that this textbook makes greater demands on the reader than do most current texts in this subject. *M. F. Smiley* (Iowa City, Iowa).

Epstein, M. P. On a class of determinants associated with a matrix. Amer. Math. Monthly **63** (1956), 160–162.

Let A be a square matrix over a field F and let

$$\Delta_m=\det[\mathrm{Tr}(A^iA^j);\ i,j=0,\cdots,m-1].$$

(1) The number, r, of distinct characteristic roots of A is the greatest integer m such that $\Delta_m\neq 0$, provided that F is suitably restricted. (2) When F is a subfield of the complex field and A is normal, replacing A^j by A^{*j} in Δ_m yields Δ_m^* and then $r+1$ is the least integer such that $\Delta_m^*=0$. [Reviewer's comments: The rule (2) holds without the * when F is formally real. The computational advantage of either (1) or (2) over the usual computation of the characteristic polynomial f and the degree of the G.C.D. of f and its derivative is exceedingly dubious.] *M. F. Smiley* (Iowa City, Ia.).

Kotelyanskiĭ, D. M. Estimates for determinants of matrices with dominant main diagonal. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 137–144. (Russian)

If we are willing to calculate the values of certain minors of a matrix A which has dominant main diagonal, we can often improve on Price's bounds for $|\det A|$ [Proc. Amer. Math. Soc. **2** (1951), 497–502; MR **12**, 793]. These new bounds are derived from new theorems on determinants of Minkowski matrices [Nachr. Ges. Wiss.

Göttingen, Math.-Phys. Kl. **1900**, 90–93], i.e. matrices with real elements in which the diagonal elements are positive; other elements, non-positive; row-sums, positive. *J. L. Brenner* (Pullman, Wash.).

van der Blij, F. A theorem on positive matrices. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 108–109.

An elementary matric proof is given of the following generalization of a result of Rédei [Math. Ann. **122** (1950), 340–342; MR **12**, 585]. Let K be a formally real field and let R be a principal-ideal subring of integral elements of K. If S is a positive symmetric n-by-n matrix over K and C is an integral n-by-m matrix ($n \geqq m$) whose maximal minors are coprime, then $G=S-SC(C'SC)^{-1}C'S$ has rank $n-m$ and there is an integral unimodular transformation which replaces $x'Gx$ by a form in $n-m$ variables whose determinant is $|S|\cdot|C'SC|^{-1}$. *M. F. Smiley.*

Hillman, A. P. On a matrix characteristic value problem. J. Aero. Sci. **23** (1956), 286.

Marcus, M. An eigenvalue inequality for the product of normal matrices. Amer. Math. Monthly **63** (1956), 173–174.

According to a theorem of G. Birkhoff [Univ. Nac. Tucumán. Rev. A. **5** (1946), 147–151; MR **8**, 561], if the elements of a square matrix are non-negative and if the sum of the elements in each row and in each column is one, then the matrix is a weighted mean of permutation matrices. Using this fact, the author proves the following result: If A, B are two normal matrices of order n, with eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$ respectively, each arranged in an arbitrary order, then the real part of the trace of AB is between the minimum and maximum of $\operatorname{Re}\sum_{i=1}^{n}\alpha_{p(i)}\beta_i$, when p runs over all permutations of $\{1, 2, \cdots, n\}$. Reviewer's observation: This result can be refined to the following form: Let A, B, $\{\alpha_i\}$, $\{\beta_i\}$ be as above. If $\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_n$ are the eigenvalues of the Hermitian matrix $\frac{1}{2}(AB+B^*A^*)$, then for any positive integer $k \leqq n$, we have

$$\sum_{i=1}^{k}\gamma_i \geqq \operatorname{Min}\operatorname{Re}\sum_{i=1}^{k}\alpha_{\nu_i}\beta_{\nu_i},\ \sum_{i=1}^{k}\gamma_{n-i+1} \leqq \max\operatorname{Re}\sum_{i=1}^{k}\alpha_{\nu_i}\beta_{\nu_i},$$

where, for both minimum and maximum $\{\nu_1, \nu_2, \cdots, \nu_k\}$ varies over all sets of k distinct positive integers $\leqq n$ in all possible arrangements. This generalized result can be derived from G. Birkhoff's theorem cited above and a theorem by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655; MR **11**, 600] on an extremum property of the sum of the first k (or the last k) eigenvalues of a Hermitian matrix. *Ky Fan* (Notre Dame, Ind.).

Moyls, B. N.; and Marcus, M. D. Field convexity of a square matrix. Proc. Amer. Math. Soc. **6** (1955), 981–983.

Let C be a complex n-squared matrix, $F(C)$ be the range of the function $f(z)=(Cz, z)$, and $P(C)$ the convex closure of the eigenvalues of C. The authors are concerned with finding necessary and sufficient conditions in order that $F(C)=P(C)$. A solution is given for triangular matrices. If $n \leqq 4$, then $F(C)=P(C)$ if and only if C is normal. If $n \geqq 5$, there exist matrices C which are not normal and for which $F(C)=P(C)$. *M. R. Hestenes.*

Dobronravov. V. V. On constructing sufficient criteria for stability. Avtomat. i Telemeh. **17** (1956), 211–216; appendix to no. 3, 1–2. (Russian. English summary)

Let y be a vector; (1) $\dot{y}+Ay=0$; $A=(a_{ij})$; $a_{ij} > \sum_{j}' |a_{ij}|$ (dominant main diagonal). Then all solutions of (1) are stable, since the characteristic roots of A have non-positive real part. Two similar theorems are given, based on Theorem I of the reviewer's paper in Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 452–454 [MR **15**, 926]. One states that if the diagonal element dominates in the last $n-2$ rows, if the absolute value of the principal minor on the first two rows exceeds the sum of the absolute values of the non-principal minors, and if $a_{11}+a_{22} > \sum_{j>2}(|a_{1j}|+|a_{2j}|)$, then A has no real positive characteristic root.

J. L. Brenner (Pullman, Wash.).

Stojaković, Mirko. Sur une propriété des matrices quasi-inverses. Bull. Soc. Math. Phys. Serbie **6** (1954), 154–157. (Serbo-Croatian summary)

The author proves two theorems announced earlier [C. R. Acad. Sci. Paris **236** (1953), 877–879; MR **14**, 1055]. Let matrix A be of order $l \times k$ and rank l ($l \leqq k$). In the author's previous terminology, a "quasi-inverse" of A is any matrix B of order $k \times l$ such that $AB=I_l$, the identity. Then $E=BA$ (of order $k \times k$ and rank l) was previously called a "quasi-unit" matrix. It is noted that $AE=A$, $EB=B$, $E^2=E$. Theorem 1. For any $i=1, \cdots, k$, the sum of all principal $i \times i$ minor determinants of E is $\binom{l}{i}$. Theorem 2. The characteristic polynomial of E is $\lambda^{k-l}(\lambda-1)^l$. In contradiction to his earlier terminology and the sense of the words, here the author calls E a quasi-inverse of A. *G. E. Forsythe.*

Terracini, Alessandro. Matrici permutabili con la propria derivata. Ann. Mat. Pura Appl. (4) **40** (1955), 99–112.

Let $A=A(t)$ be an $n \times n$-matrix whose elements $a_{ij}(t)$ are real or complex functions of the real variable t, analytic within a common interval. Let $A'=A'(t)$ be the matrix of the derivatives $a_{ij}'(t)$. It is known that from

$$(*) \qquad A(t)A'(t)=A'(t)(At)$$

follows the "finite commutativity" (f.c.): $A(t_2)A(t_1)=A(t_1)A(t_2)$ if $A(t)$ is non-derogatory. It is also known that this sufficient condition is not necessary. The author studies classes of (not necessarily) derogatory matrices A for which f.c. follows from (*) or from some additional conditions as $AA^{(h)}=A^{(h)}A$ ($h>1$). First it is shown that (*) guarantees f.c. if $A(t)$ has all the roots of its minimum polynomial simple; for in this case A' and therefore all the higher derivatives A'', A''', $\cdots$ appear as polynomials in A whose coefficients are scalar functions of t. The following theorem is the main tool for the further discussion: Let $A(t)$ be a matrix with r different characteristic roots λ_t and the Segre characteristic

$$[(n_{1,s_1}n_{1,s_1-1}\cdots n_{1,1})\cdots(n_{r,s}n_{r,s-1}\cdots n_{r,1})]$$

so that $(\lambda-\lambda_t)^{n_{t,s_t}}, \cdots, (\lambda-\lambda_t)^{n_{t,1}}$ are the elementary divisors of A for the root λ_t. Let m be the least positive integer such that $2m > \sum_{t=1}^{r}\sum_{j=1}^{s_t}(2s_t-2j+1)n_{t,j}=\sigma$. If A commutes with the matrices A', A''', $\cdots$, $A^{(2m-1)}$, then f.c. is valid. The proof uses the theorem of Frobenius according to which the exact number of linearly independent matrices commuting with A equals σ. A theorem which is weaker with regard to the number of successive derivatives of odd order, supposed to be commuting with A, can be derived without Frobenius' theorem. Next the case $[(2\,1\cdots1)]$ is discussed for $n \geqq 3$, in particular matrices $A=m(t)I+(b_i(t)\eta_j(t))_{i,j=1}^{n}$ with the condition $\sum_{i=1}^{n}b_i\eta_i=0$, so that the projectivity associated with A is a "special homology" with the centre b and the axis plane

η. Then (*) is equivalent with the condition $\sum_{i=1}^{n} b_i'\eta_i=0$ which geometrically means that the axis plane of the homology touches the curve $b(t)$ at the corresponding centre b. Conditions are given under which f.c. follows from (*). Similar results are obtained for matrices A of Segre characteristic $[(2\,2\,1\cdots1)]$ and others, and matrices of the form $A=mI+(b_i\eta_j)+(c_i\zeta_j)$ with incidence hypotheses $\sum b_i\eta_i=\sum b_i\zeta_i=\sum c_i\eta_i=\sum c_i\zeta_i=0$ are investigated. For $n=3$ the main theorem shows that f.c. follows from (*) in all cases except for the Segre characteristic $[(2\,1)]$; in this case f.c. is obtained if apart from (*) the supposition $AA'''=A'''A$ is made. In the case $n=4$ the same is shown for matrices having the Segre characteristics $[(2\,1)1]$, $[(3\,1)]$, $[(2\,1\,1)]$. The results of this paper make it evident that a detailed discussion of many particular cases cannot be avoided so that a simple comprehensive answer to the main question cannot be expected. *H. Schwerdtfeger* (Melbourne).

Spampinato, Nicolò. La risoluzione delle equazioni algebriche del secondo e terzo grado nel campo biduale e tripotenziale. Ricerca, Napoli **5** (1954), nos. 1–2, 15–26; no. 3, 12–22; no. 4, 13–17.

See also: Kazimirskiĭ, p. 821; Halezov, p. 825; Gyires, p. 830; Ankeny and Rivlin, p. 833; Ostrowski, p. 898; Marcus, p. 898; Bondar', p. 899; Bauer, p. 900.

Rings, Fields, Algebras

Tallini, Giuseppe. Sui sistemi a doppia composizione ordinati archimedei. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 367–373.

The paper is concerned with Archimedean ordered rings, not assumed to be associative. Theorem I states that such rings are both associative and commutative, while Theorem II states that such a ring, possessing an idempotent $e>0$, is order-isomorphic to a subring of the real field. The results mentioned are in the literature [see Hion, Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4(62), 237–242; MR **16**, 442]. Indeed, Hion proves Theorem II with the hypothesis of an idempotent $e>0$ replaced by the requirement that there exist elements a and b such that $ab>0$.

R. L. San Soucie (Eugene, Ore.).

Amitsur, Avraham Shimshon. The radical ring generated by a single element. Riveon Lematematika **9** (1955), 41–44. (Hebrew. English summary)

Let R be the Jacobson radical of a ring A and let a be an element of R. Using transfinite induction, the author constructs a ring A' which contains a and is radical (coincides with its Jacobson radical) and as such minimal (is a subring of any other ring with the same property). He shows that A' is commutative. Moreover, if A is an algebra over a field F then A' also is an algebra over F, and if a is algebraic with respect to F then A' is nilpotent while if a is transcendental, then A' is isomorphic to the ring $tF(t)$, where t is an indeterminate. *A. Robinson.*

Kazimirskiĭ, P. S. Compatibility conditions for a non-homogeneous system of linear equations in a noncommutative ring of principal ideals. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. **1** (1955), 45–51. (Russian)

Necessary and sufficient conditions that the simultaneous equations $Ax=b$ be solvable are: 1) rank $A=$ rank of the augmented matrix (Ab); 2) the elementary divisors of these matrices are similar. Jacobson's book [Theory of rings, Amer. Math. Soc. Math. Surveys, v. 1, New York, 1943; MR **5**, 31] is cited in the bibliography.

J. L. Brenner (Pullman, Wash.).

Tominaga, Hisao. On a theorem of N. Jacobson. Proc. Japan Acad. **31** (1955), 653–654.

Let R be a simple ring with the descending-chain condition containing a simple subring R' such that R' is finite dimensional over its centre and of characteristic $\neq2$. The author proves that if the (left) dimension of R over R' is 2 then R is a Galois extension of R'. This extends a result of N. Jacobson [Amer. J. Math. **77** (1955), 593–599; MR **17**, 9] who proved this fact for division rings. The extension is obtained by similar methods with the use of Nakayama's Galois theory for simple rings [Trans. Amer. Math. Soc. **73** (1952), 276–292; MR **14**, 240].

S. A. Amitsur (Jerusalem).

Kertész, Andor. A new characterization of semi-simple rings. Acta Univ. Debrecen. **1** (1954), 151–153; additamentum ad **1** (1955), 21. (Hungarian. English summary)

By a semi-simple ring (in the classical sense) we mean a ring containing no nilpotent left ideal and satisfying the descending-chain condition for the left ideals. The importance of this category of rings is shown not only by the well-known classical structure theorems, but also by the fact that the semi-simple rings as domains of operators for modules have remarkable characteristic properties. The present note contains such a new characterization of semi-simple rings, namely the following theorem: An arbitrary ring R is semi-simple if and only if for an arbitrary left ideal L of R, and for an arbitrary element g of an arbitrary left R-module G the submodule Lg is a direct summand of G. From this theorem the following theorem of O. Goldman easily follows [Bull. Amer. Math. Soc., **52** (1946), 1021–1027; MR **8**, 433]: An arbitrary ring R is semi-simple if and only if any R-module is the direct sum of the submodule annihilated by R and of minimal submodules. *Author's summary.*

Brown, William P. The semisimplicity of ω_f^n. Ann. of Math. (2) **63** (1956), 324–335.

Soit P un espace vectoriel de dimension n sur un corps de caractéristique 0, et soit P_f l'espace des tenseurs (contravariants) d'ordre f sur P. Le groupe symétrique $\mathfrak{S}_f$ et le groupe linéaire $\mathrm{GL}(n)$ ont des représentations canoniques dans l'algèbre Ω des endomorphismes de P_f, et on sait depuis I. Schur que les algèbres engendrées par les images de ces deux groupes sont commutantes l'une de l'autre. Considérons de même l'image canonique du groupe orthogonal $\mathrm{O}(n)$ dans Ω; R. Brauer a montré que l'algèbre engendrée par cette image est la commutante de l'image d'une algèbre ω_f^n dont il a donné une définition combinatoire au moyen de diagrammes [Ann. of Math. (2) **38** (1937), 857–872]. L'image canonique de ω_f^n est donc semi-simple, mais la question de savoir s'il en était de même de ω_f^n restait ouverte. L'auteur résout cette question en montrant que ω_f^n est semi-simple si et seulement si $n\geq f-1$. Pour cela, il introduit, avec Brauer, une suite décroissante $A_0=\omega_f^n\supset A_1\supset\cdots\supset A_m$ d'idéaux bilatères de ω_f^n (avec $m=[\frac{1}{2}f]$, et se ramène à prouver que A_r est semi-simple si et seulement si $n\geq f-r$. Il commence par montrer, par une étude soignée de la représentation définie par Brauer, que la condition $n\geq f-r$ est nécessaire et suffisante

pour que A_r soit représenté fidèlement dans Ω; ceci établit aussitôt que si $n \geqq f-r$, A_r est semi-simple. Pour démontrer la réciproque, il se ramène aux deux cas $n=f-r-1$ et $r<m$, $n<f-m$ et $r=m$; dans le second cas, il remarque que A_m ne peut être semi-simple que s'il est simple (en raison de la présence d'un idempotent e tel que eA_me soit un corps et $A_meA_m=A_m$); or le noyau de la représentation de A_m dans Ω n'est pas nul et on voit assez facilement qu'il n'est pas A_m tout entier, donc A_m ne peut être semi-simple. La méthode est analogue dans le premier cas, en décomposant A_r en somme directe de A_{r+1} et d'idéaux bilatères qui ne peuvent être semi-simples que s'ils sont simples, et en prouvant que cela ne peut avoir lieu en raison de $n<f-r$. *J. Dieudonné.*

Robinson, Abraham. On ordered fields and definite functions. Math. Ann. **130** (1955), 257–271.

Eine Aussage

$$X=(x_1)\cdots(x_m)(\exists y_1)\cdots(\exists y_n)Z(x_1\cdots x_m; y_1\cdots y_n)$$

des (elementaren) Prädikatenkalküls, in der Z quantorenfrei ist, heisse ein (AE)-Axiom. Verf. beweist zunächst, dass zu einem Modell M eines (AE)-Axiomensystems S allemal Erweiterungen von M existieren, die ausser S noch X erfüllen, wenn nur für alle $a_1, \cdots, a_m$ aus M Erweiterungen von M existieren, die ausser S noch

$$(\exists y_1)\cdots(\exists y_n)Z(a_1, \cdots, a_m; y_1, \cdots, y_n)$$

erfüllen. Diesen Satz wendet Verf. auf formal-reelle Körper und positiv-definite Funktionen an. Er erhält dabei die bekannten Ergebnisse von Artin, gelangt aber auch zu Verallgemeinerungen und Verschärfungen, die ohne die metamathematischen Hilfsmittel nicht erreichbar zu sein scheinen. *P. Lorenzen* (Bonn).

Amitsur, A. S. Algebras over infinite fields. Proc. Amer. Math. Soc. **7** (1956), 35–48.

For an algebra A over a field F satisfying $(A:F)<$ card F it is proved that (1) the quasi-regular one-sided ideals of A are its nil one-sided ideals, (2) every right nil ideal is contained in a two-sided nil ideal of A, and (3) the union of right nil ideals in A is a nil right ideal contained in $K(A)$, the maximal two-sided nil ideal of A. The third result above gives an affirmative answer to a problem of Köthe [Math. Z. **32** (1930), 161–186] for algebras of relatively low order and it is later extended to hold for arbitrary algebras over nondenumerable fields. The first result mentioned implies that the Jacobson radical of a finitely generated algebra over a nondenumerable field is its nil radical. The proofs are made by studying properties of the sets $C_F(a)$, which consist of all λ in F for which $-\lambda^{-1}a$ is quasi-regular, a in A. An algebra A is called an LBI(LBD) algebra if every finitely generated submodule of A consists of nilpotent (algebraic) elements of bounded index (degree). Every nil (algebraic) algebra over a nondenumerable field is an LBI (LBD) algebra. Among the results obtained from a study of the properties LBI and LBD is that if A is an algebraic algebra over a nondenumerable field F and H an extension of F, then every extension algebra A_H and matrix algebra A_n are algebraic algebras. This gives a solution for algebras over nondenumerable fields of a problem proposed by Jacobson [Amer. J. Math. **67** (1945), 300–320; MR **7**, 2].

L. A. Kokoris (St. Louis, Mo.).

Jacobson, N. A theorem on the structure of Jordan algebras. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 140–147.

Let A be a Jordan algebra of characteristic $\neq 2$. The author introduces a ternary composition $\{abc\}$ in A defined by $2\{abc\}=(ab)c+(bc)a-(ac)b$. For a in A he defines an operator U_a on A by $bU_a=\{aba\}$. If A has an identity u, then a is defined to be regular in case there exists b in A satisfying $ab=4u$, $a^2b=4a$, $a^2b^2=16u$; b is called an inverse of a. If a is regular, it is shown that U_a is regular. It follows that the inverse b is unique. Also if z is nilpotent, then $\{zaz\}$ is not regular for any a in A.

Inverses and identities for ternary composition (see the following review concerning the identity $\{aba\}^2=\{a\{ba^2b\}a\}$) are used in the proof of the following generalization of a theorem of Albert. Let A be a finite-dimensional Jordan algebra with identity u over a field Φ of characteristic $\neq 2$, and suppose that every element of A has the form $\alpha u+z$ with $\alpha \in \Phi$ and z nilpotent; then $A=\Phi u+B$ where B is a nil subalgebra of A. It follows that any simple Jordan algebra over an algebraically closed field Φ having only one nonzero idempotent is necessarily Φu. This completes Albert's classification [Trans. Amer. Math. Soc. **69** (1950), 503–527; MR **12**, 475] of simple Jordan algebras and the author's theorems on representations of finite-dimensional separable algebras [Osaka Math. J. **6** (1954), 1–71; MR **16**, 330].

R. D. Schafer (Storrs, Conn.).

Harper, Laurence R., Jr. Proof of an indentity on Jordan algebras. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 137–139.

Let A be a Jordan algebra of characteristic $\neq 2$, and $2\{abc\}=(ab)c+(bc)a-(ac)b$ for a, b, c in A. Jacobson conjectured the identity $\{aba\}^2=\{a\{ba^2b\}a\}$, which is proved in this paper and used in a crucial way in the paper reviewed above. Marshall Hall has obtained independently a proof of the identity. *R. D. Schafer* (Storrs, Conn.).

Širšov, A. I. On special J-rings. Mat. Sb. N.S. **38(80)** 1956), 149–166. (Russian)

Theorem: the free Jordan algebra on two generators is special. This important result allows one to check mechanically any proposed identity for two elements of a Jordan algebra. A certain identity played a key role in Jacobson's completion of the structure theory, and was verified by L. Harper [see the paper reviewed above] and independently by M. Hall.

The main lemma, proved by a prodigious computation, asserts that a certain mapping $*$ defined on the free associative algebra on two generators (a and b) is a homomorphism relative to the Jordan $\circ$ operation. Definition of $*$: (1) $t^*=t$ if t is a power of a or b; (2) $t^*=a^s\circ b^r$ if t is a^sb^r or b^ra^s; (3) $t^*=a^m\circ(ca^n)^*+(a^mc)^*\circ a^n-c^*\circ a^{m+n}$ if $t=a^mca^n$; (4) $t^*=2a^m\circ(b^n\circ c^*)+2b^n\circ(a^m\circ c^*)-2(a^m\circ b^n)\circ c^*-(b^nca^m)^*$ if $t=a^mcb^n$. In part (4) the word c begins with b and ends with a, so that the definition is genuinely inductive. The paper concludes with some remarks on the generalization to rings with operators. *I. Kaplansky.*

Gravett, K. A. H. Note on a result of Krull. Proc. Cambridge Philos. Soc. **52** (1956), 379.

The author gives a short and simple proof of the following result of Krull [J. Reine Angew. Math. **167** (1932), 160–196]: Let K be a field with valuation v, value group Γ and residue class field R; let $|S|$ denote the cardinal number of the set S; then $|K| \leqq |R|^{|\Gamma|}$. *B. N. Moyls.*

See also: Brauer, p. 824; Etherington, p. 825; Newman and Taussky, p. 829; Sce, p. 837; Moisil, p. 837; Patterson, p. 876.

Groups, Generalized Groups

★**Carmichael, Robert D. Introduction to the theory of groups of finite order.** Dover Publications, Inc., New York, 1956. xiv+447 pp. Paperbound: $2.00; clothbound: $3.95.

Reprint by photo-offset of the edition of 1937 [Ginn, Boston].

Mal'cev, A. On a general method for obtaining local theorems in group theory. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. **1** (1941), no. 1, 3–9. (Russian)

This paper (published in 1941) concerns the application of a theorem in logic to group theory. The theorem asserts (roughly) that a set of propositions is consistent if every finite subset is consistent, and is quoted from an earlier paper of the author [Mat. Sb. N.S. **1(43)** (1936), 323–335]. Let $E_1, \cdots, E_k$ be "elementary" properties of a group. Say that a group G has type $[E_1, \cdots, E_k]$ if it possesses a normal series $G \supset G_1 \supset \cdots \supset G_k$ such that each G_i/G_{i+1} has property E_{i+1}. Theorem: G is of type $[E_1, \cdots, E_k]$ if and only if this is true for each finitely generated subgroup. An interesting application is the case where $k=2$ and E_2 is "abelian", E_1 is "order $\leqq m$ for a certain fixed m". In this way we can pass from Jordan's theorem that any finite group of complex n-by-n matrices has an abelian subgroup of index $\leqq$ a number m depending only on n, to Schur's theorem that the same is true for any (possibly infinite) torsion group of matrices. Further applications are given to a theorem of Černikov [ibid. **8(50)** (1940), 377–394; MR **2**, 217] on Sylow series and a theorem attributed to Baer on lattice isomorphisms of groups.

I. Kaplansky (Chicago, Ill.).

Rühs, F. Über ein spezielles Rédeisches schiefes Produkt in der Gruppentheorie. Acta Sci. Math. Szeged **16** (1955), 160–164.

The skew product of two groups G, Γ has been introduced by Rédei [J. Reine Angew. Math. **188** (1950), 201–227; MR **14**, 13]; further cases of skew products have been treated by Kochendörffer [ibid. **192** (1953), 96–101; MR **15**, 852] and by Rédei and Stöhr [Acta Sci. Math. Szeged **15** (1953), 7–11; MR **15**, 853]. In the present note the author deals with the remaining case where the product of pairs is defined by the rule $(a, \alpha)(b, \beta) = (a^{\beta}b^{\alpha}, \alpha^{b}\beta^{a})$. He proves that this yields essentially only the direct product of G and Γ. Indeed, after a suitable "permutation" Π defined by Π: $(a, \alpha) \Rightarrow (g^{-1}ah, \gamma^{-1}\alpha\eta)$, $(g, h \in G;\ \gamma, \eta \in \Gamma)$ the unit element of the skew product will be $(\varepsilon, \varepsilon)$, where ε, ε are the unit elements of G, Γ respectively, and then it follows easily that the resulting group is the direct product of groups isomorphic to G and Γ. *K. A. Hirsch.*

Rédei, L.; und Szép, J. Die Verallgemeinerung der Theorie des Gruppenproduktes von Zappa-Casadio. Acta Sci. Math. Szeged **16** (1955), 165–170.

Given are two groups G, Γ. The authors wish to describe all the groups $\mathfrak{G} = G'\Gamma'$ that are products of two subgroups isomorphic to G and Γ, respectively. The case where the intersection $G' \cap \Gamma'$ is a normal subgroup, in particular the unit subgroup, of $\mathfrak{G}$ has been treated by Zappa [Atti 2º Congresso Un. Mat. Ital., Bologna, 1940, Edizioni Cremonese, Rome, 1942, pp. 119–125; MR **8**, 367] and Casadio [Rend. Mat. e Appl. (5) **2** (1941), 348–360; MR **8**, 251; **10**, 855]. The authors treat the most general case. Their solution is however to unwieldy to be reproduced here. *K. A. Hirsch* (London).

Ōhara, Akiko. Note on commutator subgroups of factorisable groups. Proc. Japan Acad. **31** (1955), 612–614.

Let $G=AB$ be a group that is factorized by two subgroups A and B and let $[A, B]$ be the commutator group of A and B in G, generated by all the commutators $[a, b]$ with $a \in A$, $b \in B$. The main result of the author is that $[A, B]$ is normal in G. [Reviewer's remark: this result is true for every group G that is generated by two subgroups $G=\{A, B\}$. Indeed for $\alpha \in A$ we have $\alpha^{-1}[a, b]\alpha = [a\alpha, b][\alpha, b]^{-1} \in [A, B]$ and for $\beta \in B$ similarly $\beta^{-1}[a, b]\beta = [a, \beta]^{-1}[a, b\beta] \in [A, B]$, and since every element of G is a finite product of elements of A and B, the group $[A, B]$ contains together with each element all its conjugates in G.] Simple corollaries of the theorem are: 1) When $G=AB$ with A and B abelian, then G is soluble of derived length at most 2. 2) When $G=AB$ with A abelian and B soluble of derived length 2, and if G is soluble, then the derived length of G exceeds that of $[A, B]$ by at most 2. [see also N. Itô, Math. Z. **62** (1955), 400–401; MR **17**, 125.]

K. A. Hirsch (London).

Szép, J.; und Itô, N. Über die Faktorisation von Gruppen. Acta Sci. Math. Szeged **16** (1955), 229–231.

The main result of the note is the theorem that every finite group G that is not nilpotent contains a nilpotent subgroup H whose normal closure $\overline{H}$ is G. The simple proof, by induction on the order, makes use of the fact that a finite group in which all maximal subgroups are normal is nilpotent. The normal closure $\overline{H}$ of a subgroup H of a finite group G is, of course, the product of H with all its conjugate subgroups in G. This characterisation is extended to arbitrary groups, where the number of conjugates of H may be infinite. *K. A. Hirsch*

Amato, Vincenzo. Gruppi diedrali e tipi dei gruppi non abeliani finiti. Matematiche, Catania **10** (1955), 149–152.

Smirnov, D. M. Infrainvariant subgroups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 92–96. (Russian)

Detailed proofs of some results already announced [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 643–646; MR **12**, 587]. *I. Kaplansky* (Chicago, Ill.).

Vinogradov, A. A. Partially ordered locally nilpotent groups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 3–18. (Russian)

The paper is concerned with the effect of nilpotency assumptions on a partially ordered group. The four theorems of § 1 read as follows. 1. If a partially ordered group G has an ascending central series of convex subgroups with all quotient groups torsion-free, then the ordering on G can be strengthened to a simple order. (This was known previously for abelian groups). 2. If G is a nilpotent torsion-free partially ordered group which is nilpotent of class two, then G can be embedded in a divisible torsion-free partially ordered group H which is nilpotent of class two. In H, $a^n > 1$ implies $a > 1$. 3. If a divisible partially ordered group G is nilpotent of class two and has the property that $a^n > 1$ implies $a > 1$, then G possesses a proper normal convex subgroup. 4. If a torsion-free partially ordered group G is nilpotent of class 2, then the ordering on G can be strengthened to a simple order. The second section is devoted to lattice-ordered groups. Theorem 5 asserts that a lattice-ordered group with no proper convex subgroups is a subgroup of the real numbers. The proof is obscure to the reviewer. The

last four theorems depend on Theorem 5. The final theorem provides a partial solution of a problem posed by Birkhoff [Ann. of Math. (2) **43** (1942), 298–331; MR **4**, 3]. If every finite subset of a lattice-ordered group G lies in a nilpotent l-subgroup, then G has proper l-ideals.

I. Kaplansky (Chicago, Ill.).

Golovin, O. N. Nilpotent products of groups. Amer. Math. Soc. Transl. (2) **2** (1956), 89–115.

Translated from Mat. Sb. N.S. **27(69)** (1950), 427–454; MR **12**, 672.

Golovin, O. N. Metabelian products of groups. Amer. Math. Soc. Transl. (2) **2** (1956), 117–131.

Translated from Mat. Sb. N.S. **28(10)** (1951), 431–444; MR **13**, 105.

Golovin, O. N. On the problem of isomorphisms of nilpotent decompositions of a group. Amer. Math. Soc. Transl. (2) **2** (1956), 133–140.

Translated from Mat. Sb. N.S. **28(70)** (1951), 445–452; MR **13**, 105.

Mal'cev, A. I. On certain classes of infinite solvable groups. Amer. Math. Soc. Transl. (2) **2** (1956), 1–21.

Translated from Mat. Sb. N.S. **28(70)** (1951), 567–588; MR **13**, 203.

Kulikov, L. Ya. On direct decompositions of groups. Amer. Math. Soc. Transl. (2) **2** (1956), 23–87.

Translated from Ukrain. Mat. Ž. **4** (1952), 230–275, 347–372; MR **15**, 396.

Brenner, Joël Lee. Quelques groupes libres de matrices. C. R. Acad. Sci. Paris **241** (1955), 1689–1691.

The author proves that the group generated by the two matrices

$$A=\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}$$

is a free group for all real $m\geqq 2$. The elements of the group are, of course, of the form

$$M=\begin{pmatrix}1+k_{11}m^2 & k_{12}m\\ k_{21}m & 1+k_{22}m^2\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

with integers k_{ij} and determinant 1. The case when $n=2$ has been treated by Sanov [Doklady Akad. Nauk SSSR (N.S.) **57** (1947), 657–659; MR **9**, 224] who showed that, conversely, every matrix of the form M belongs to the group. In the general case this is no longer true. The additional condition to be satisfied by a matrix M to belong to the group is that the ratio $|a_{11}/a_{12}|$ should not lie between the roots of the equation $x^2-mx+1=0$. For $m=1$ the group in question is the full unimodular group with integer coefficients and is, therefore, not free. There is an infinity of values m between 0 and 2 for which the group is not free. The interesting question arises, therefore, whether there is an algebraic number m, $0<m<2$, for which the group turns out to be free. *K. A. Hirsch.*

Brauer, Richard. Zur Darstellungstheorie der Gruppen endlicher Ordnung. Math. Z. **63** (1956), 406–444.

The representation theory of a finite group $\mathfrak{G}$ over a given field Ω can be studied directly or via the representation theory of the corresponding group algebra Γ. After two introductory sections in which both these points of view are adopted, the author gives in §§ 3, 4 a brief but elegant survey of the necessary modular representation theory. Since two ordinary representations of $\mathfrak{G}$ belong to the same p-block if and only if their class multipliers are congruent (mod p), the algebra $Z=Z(G, \Omega)$ of the classes $\mathfrak{K}_i$ of $\mathfrak{G}$ is fundamental in this theory.

If l is the number of p-regular classes of $\mathfrak{G}$, and t is the number of p-blocks B_τ ($\tau=1, 2, \cdots, t$), let S_τ be the set of l_τ indices j for which the modular character φ_j belongs to B_τ and $l=l_1+l_2+\cdots+l_t$. Then the ordinary characters χ_i and the classes $\mathfrak{K}_i$ of $\mathfrak{G}$ may be so enumerated that all $\mathfrak{K}_i$ with $i \in S_\tau$ are p-regular and χ_i also belongs to B_τ. With such an enumeration (Theorem 5A): $\det(\chi_i(G_j))\not\equiv 0 \pmod p$, $\det(\varphi_i(G_j))\not\equiv 0 \pmod p$, $\det(d_{ij})\not\equiv 0 \pmod p$ for $i, j \in S_\tau$. In the special case $\mathfrak{G}=S_n$ this enumeration has been given explicitly [Robinson and Taulbee, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 596–598; MR **17**, 126].

The notion of the defect of a block is introduced in § 6, and in § 7 the main idea of the paper, namely the establishment of a relationship between the block structure of any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ and that of $\mathfrak{G}$, is developed (Theorem 7E). That this relationship implies an inclusion relationship between the corresponding defect groups is proved in Theorem 8D. In particular, if $\mathfrak{H}$ is an invariant p-subgroup of $\mathfrak{G}$ of order p^μ, then every block B of $\mathfrak{G}$ of defect d contains at least one block $\bar{B}$ of $\bar{\mathfrak{G}}=\mathfrak{G}/\mathfrak{H}$ of defect $\bar{d}=d-\mu$, and every block of $\bar{\mathfrak{G}}$ contained in B has defect at most $d-\mu$ (Theorem 9E).

Perhaps the most important theorem is 10B which runs as follows: If $\mathfrak{D}$ is any subgroup of order p^d of $\mathfrak{G}$, and $\mathfrak{H}$ is the normaliser of $\mathfrak{D}$, then there is a 1-1 correspondence between the blocks B of $\mathfrak{G}$ with defect group $\mathfrak{D}$ and the blocks B of $\mathfrak{H}$ with defect d.

Congruence properties for characters modulo a certain power of p are derived in § 12. Finally the author raises the question of determining the number or blocks of given defect d. A sequel is promised. *G. de B. Robinson.*

Melvin, M. A. Simplification in finding symmetry-adapted eigenfunctions. Rev. Mod. Phys. **28** (1956), 18–44.

A review article devoted to the problem of resolving a function into its "Fourier expansion" according to the irreducible representations of a given transformation group. The method is widely used for the solution of differential equations which are known to be invariant under a group of transformations. The formal theory for finite groups has been discussed by E. Wigner [Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, Vieweg, Braunschweig, 1931, ch. 12]. The groups considered here are the finite subgroups of the rotation groups in two and three dimensions, with and without improper elements (reflections). The paper is of interest primarily for its applicability to special problems such as those of molecular and crystal structure. The mathematical difficulties which arise in the extension to more complicated (non-compact) groups, such as the Lorentz group, are not discussed. *E. L. Hill.*

Roman, Tiberiu. Le système des générateurs des groupes abstraits, correspondantes aux 32 classes cristallographiques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 9–21. (Romanian. Russian and French summaries)

Geometrische Diskussion der 32 kristallgraphischen Klassen mit Angabe ihrer sämtlichen Untergruppen. Ferner wird für jede Klasse ein System von erzeugenden Operationen angegeben. In einer Tabelle wird die Ok-

taedergruppe O_h mit ihren sämtlichen Untergruppen dargestellt. Die diesbezügliche Bemerkung des Verf., dass die entsprechende Darstellung des Ref. [Die Bewegungsgruppen der Kristallographie, Birkhäuser, Basel, 1947, S. 72; MR **8**, 562] unvollständig sei, beruht auf einer irrtümlichen Interpretation der dort angegebenen Tabellen: Die Klasse D_{4h} ist dort gesondert dargestellt, um eine bessere Uebersicht zu gewinnen.

J. J. Burckhardt (Zürich).

Fletcher, T. J. Film groups. Math. Gaz. **40** (1956), 15–19.

Halezov, E. A. Isomorphisms of matrix semigroups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 42–56. (Russian)

Let F_n^r be the multiplicative semigroup of all square matrices with elements in the (non-commutative) field F having order n and rank not exceeding r. If the semigroup F_n^r, where $n \geqq 2$ and $1 \leqq r \leqq n$, is isomorphic to the semigroup G_m^s, then $r=s$ and $n=m$ and F is isomorphic to the field G. Every such isomorphism θ of F_n^r onto G_n^r has the form $\|x_{ij}\|^\theta = S\|x_{ij}^\phi\|S^{-1}$, where ϕ is a particular isomorphism of the field F onto the field G and where S is a certain non-singular nth order matrix over G.

R. A. Good (College Park, Md.).

Lyapin, E. S. Potential inversion of elements in semigroups. Mat. Sb. N.S. **38(80)** (1956), 373–388. (Russian)

An element X of a semigroup G is called invertible if for all $A \in G$ there exists elements Y and $Z \in G$ such that $YX = A$, $XZ = A$; X is called potentially invertible if there exists a semigroup $G^* \supset G$ such that the element X is an invertible element in G^*. The author proves: In order that an element X of a semigroup G be potentially invertible, it is necessary and sufficient that for all $A \in G$, the equations $XS = A$ and $S'X = A$ have no more than one solution S and $S' \in G$. Examples from semigroups of infinite matrices are given.

L. J. Paige.

Mal'cev, A. I. Nilpotent semigroups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 107–111. (Russian)

The author showed [Mat. Sb. N.S. **6(48)** (1939), 331–336; MR **2**, 7] that a cancellation semigroup need not be embeddable in a group. The point of the present paper is to show that embedding is possible for a nilpotent cancellation semigroup (and in a minimal embedding the group just consists of the elements $g_1 g_2^{-1}$). Of course one must define nilpotence in a fashion meaningful for semigroups. For elements x, y, u_1, $\cdots$, u_n define $X_0 = x$, $Y_0 = y$, $X_{n+1} = X_n u_{n+1} Y_n$, $Y_{n+1} = Y_n u_{n+1} X_n$. Then a semigroup is said to be nilpotent (of class $\leq n$) if one always has $X_n = Y_n$.

I. Kaplansky (Chicago Ill.).

Vandiver, H. S.; and Weaver, M. W. A development of associative algebra and an algebraic theory of numbers. III. Math. Mag. **29** (1956), 135–151.

The first two papers with the above title were written by Vandiver [Math. Mag. **25** (1952), 233–250; **27** (1953), 1–18; MR **14**, 348; **15**, 202]; the present paper can be read independently of its predecessors. The paper deals mainly with the definition and elementary theory of semi-groups as an example of a simple algebraic system. An account is also given of what the authors call a "correspondence", this being a mapping of a finite set into itself. For example, the factoring of a correspondence into "excycles" (a generalization of the familiar cycle in the case of substitutions) is discussed. In this connection the main result (due to Stoll) is that any finite semi-group is isomorphic to some semi-group of correspondences, a generalization of Cayley's theorem for substitution groups. Many examples are given and there are problems for the reader to do; included among these problems are most of the proofs of the theorems given in the paper.

H. W. Brinkmann.

Etherington, I. M. H. Theory of indices for non-associative algebra. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954–55), 150–160 (1956).

The author's previous definition of the logarithmetic of an algebra [J. London Math. Soc. **16** (1941), 48–55; and Proc. London Math. Soc. (2) **52** (1951), 241–252; MR **3**, 103; **12**, 798] is generalized so that not only is every logarithmetic a homomorphic image of the free additive groupoid B with one generator, but every homomorphic image of B becomes a logarithmetic in the wider sense.

An algebra G is a system with a binary multiplication which may be non-commutative and non-associative. If $x \in G$, corresponding to every power of x there is an index A which depends not only on the number of x's in the power but also on the manner in which they are bracketed. The sum A+B of two indices is defined by $x^{A+B} = x^A x^B$ so that every index is a sum of 1's suitably bracketed. Let S be any set of elements drawn from one or more algebras. If $x \in S$ and A is any index, x^A is defined but not necessarily in S. Indices A and B are equal relative to S if $x^A = x^B$ for all x in S. The class of indices equal to A relative to S is denoted by N_A and is called an (index) number. The sum $N_A + N_B$ of two index numbers is unambiguously defined as N_{A+B}. With the sum so defined the numbers N_A constitute the logarithmetic of S, denoted by $\log S$. Denote by $\bar{S}$ the set consisting of all powers of elements of S. The closure of $\log S$ is defined to be $\log \bar{S}$. It is shown that $\log \bar{S} = \log \log S$ so that the closure of $\log S$ depends only on $\log S$ and not on the particular S which gives rise to $\log S$. If $\log S = \log \bar{S}$, $\log S$ is closed. A logarithmetic is closed if and only if it is the logarithmetic of an algebra or a class of algebras. Multiplication of indices is defined by $x^{AB} = (X^A)^B$ and in a closed logarithmetic this induces a multiplication of index numbers defined by $N_A N_B = N_{AB}$. There is a brief discussion of logarithmetics which satisfy the commutative and associative laws (of addition), the entropic law, (A+B)+(C+D)=(A+C)+(B+D), and the palintropic law, AB=BA.

D. C. Murdoch (Vancouver, B.C.).

Thierrin, Gabriel. Sur quelques décompositions des groupoïdes. C. R. Acad. Sci. Paris **242** (1956), 596–598.

A groupoid G is g-indecomposable if it is not a union of disjoint subgroupoids. It is prime if $xy \in G$ implies either $x \in G$ or $y \in G$. Every groupoid is uniquely expressible as a union of disjoint g-indecomposable subgroupoids each of which is a maximal g-indecomposable subgroupoid of G. Every prime subgroupoid of G is a union of maximal g-indecomposable subgroupoids of G. Every groupoid is a union of disjoint subgroupoids in such a way that the equivalence ϱ^* corresponding to this decomposition is regular and the quotient groupoid G/ϱ^* is a semi-lattice homomorphic to G. A groupoid is s-indecomposable if the only semi-lattice to which G is homomorphic consists of a single element. It then follows that G is s-indecomposable if and only if it contains no proper prime ideal.

D. C. Murdoch (Vancouver, B.C.).

Dicman, A. P. On multigroups whose elements are subsets of a group. Moskov. Gos. Ped. Inst. Uč. Zap. **71** (1953), 71–79. (Russian)

A multigroup is assumed throughout to possess an identity e. A multigroup z is called an ultragroup provided z has a finite descending series of submultigroups from z to e such that each right factor is a group; if each right factor is an abelian group, the ultragroup is called abelian. Suppose a group G is partitioned into classes such that (1) the identity of G forms a class by itself, (2) if K is any class, the inverses of all the elements in K form a class, (3) the classes form a multigroup $\bar{G}$ under the operation of multiplication of complexes in G. These so-called multiclasses generalize the classes of conjugate elements. The multigroup $\bar{G}$ is a group if and only if each multiclass contains a single element. A group G is solvable if and only if it is partitionable into multiclasses so that the corresponding $\bar{G}$ is an abelian ultragroup. A group G is nilpotent if and only if the multigroup $\bar{G}$ whose elements are conjugate classes is an ultragroup. In an earlier paper by the author [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **49** (1946), 315–317; MR **7**, 511], this last condition was necessary and sufficient that G be solvable. *R. A. Good* (College Park, Md.).

See also: Brown, p. 886.

THEORY OF NUMBERS

on order

★ **Winogradow, I. M. Elemente der Zahlentheorie.** Verlag R. Oldenbourg, München, 1956. viii+156 pp. DM 10.50.

Translation of Vinogradov's Osnovy teorii čisel [6th ed., Gostehizdat, Moscow, 1952; MR **15**, 601].

Sáenz García, Clemente. Peculiarities of the system of binary enumeration. An. Asoc. Españ. Progr. Ci. **20** (1955), 535–547. (Spanish)

Mordell, L. J. Some Diophantine inequalities. Mathematika **2** (1955), 145–149.

A very general theorem is proved, of which the following is a more or less typical consequence: Let $f(x)=|x_1,\cdots x_n|g(x)$, where

$$g(x)=\sum_{r=1}^{n}\sum_{s=1}^{n}a_{rs}x_rx_s+\sum_{r=1}^{n}a_rx_r+a_0.$$

Suppose that $g(x)>0$ for all points x with coordinates numerically smaller than 1, and that $a_{rr}\geqq 0$ for $r=1,\cdots,n$. Then there is a point x congruent to any preassigned point (mod 1), such that

$$f(x)\leqq\frac{1}{2^{n+2}}\sum_{r=1}^{n}a_{rr}+\frac{1}{2^n}a_0.$$

The latter theorem generalizes a result (not yet published) of Birch and Swinnerton-Deyer. *W. J. LeVeque.*

Smiley, M. F. On the zeros of a cubic recurrence. Amer. Math. Monthly **63** (1956), 171–172.

The cubic recurrence $(T)=(T_1, T_2, T_3, \cdots)$ is defined by real numbers T_0, T_1, T_2 and

$$T_{n+3}=PT_{n+2}-QT_{n+1}+RT_n \quad (n=0, 1, 2, \cdots),$$

where P, Q, R are real. It is assumed that the roots, u v, w of z^3-Pz^2+Qz-R are nonzero, real and have distinct absolute values. If $T_k=0$, k is called a zero of (T). The following theorem is proved. The cubic recurrence (T) has at most three zeros. Ward [same Monthly **62**(1955), 155–160; MR **16**, 675] had proved this result under the restriction that u, v, w were integers relatively prime in pairs. The proof of the theorem of the paper makes use of Descartes' rule of signs. *L. Carlitz.*

Ward, Morgan. The mappings of the positive integers into themselves which preserve division. Pacific J. Math. **5** (1955), 1013–1023.

Let L denote the lattice of the integers 0, 1, 2, $\cdots$ partially ordered by division. Consider the mappings ϕ of L into itself that preserve division, that is (i) if $n|m$, then $\phi_n|\phi_m$. A mapping that satisfies (i) and (ii) $\phi_0=0$, $\phi_1=1$ is called a divisibility sequence on L; if it also satisfies (iii) $\phi_n>0$ for $n>0$, it is called normal. If $\phi_r\equiv 0$ (mod m) for some $r>0$, m is a divisor of ϕ and r a place of apparition of m in ϕ. If in addition $\phi_s\not\equiv 0$ (mod m) for every proper divisor s of r, r is called a rank of apparition of m in ϕ. If m is not a divisor of ϕ, it has rank of apparition zero. Thus every m has at least one rank of apparition in ϕ; if each m has exactly one rank of apparition, then ϕ is said to admit a rank function. Moreover (iv) $\phi_n\equiv 0$ (mod m) if and only if $n\equiv 0$ (mod ϱ_m), where ϱ_m is the rank of m in ϕ. The main results of the paper are contained in the following two theorems.

I. A necessary and sufficient condition that a normal mapping ϕ admit a rank function is that $(\phi(pn), \phi(qn))=\phi(n)$, where p and q are any distinct primes.

Let $n=p_1^{a_1}\cdots p_k^{a_k}$ and define

$$\Psi(n)=\phi(n)/\text{L.C.M.}_{i=1,\cdots,k}\,\phi(n/p_i);$$

then Ψ is called a generator of ϕ. Also define

$$\zeta(n)=\prod_{d|n}(\phi(n/d))^{\mu(d)};$$

if $\zeta(n)$ is integral for every n, ζ is called a Dedekind generator of ϕ.

II. If ϕ is a normal sequence, then a necessary and sufficient condition that ϕ admit a rank function is that its Dedekind generator should exist and be equal to its ordinary generator.

We remark that the paper makes free use of lattice terminology and notations. *L. Carlitz.*

Paasche, Ivan. Eine Verallgemeinerung des Moessnerschen Satzes. Compositio Math. **12** (1956), 263–270.

The theorem of Moessner [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 29; MR **14**, 353] is generalized as follows: Let $K_1\leqq K_2\leqq\cdots\leqq K_n$ and consider the numbers $A_n^{\lambda}{}_{,\nu}$ $(n=1, 2, \cdots; \lambda, \nu=1, 2, \cdots, K_n;\ \lambda+\nu\leqq K_n)$ as determined by the recursion formulas

$$A^{\lambda}_{n,\nu+1}=A^{\lambda}_{n\nu}+A^{\lambda-1}_{n,\nu+1},$$

$$A^{\lambda}_{n+1,1}=A^{\lambda}_{n,1-\lambda+K_n}+A^{\lambda-1}_{n+1,1},$$

$$A^{0}_{n\nu}=A_{n\nu},\quad A^{\lambda}_{11}=A^{0}_{11}$$

from the initial values $A_{n\nu}$. The determination of these numbers is accomplished by means of a certain set of generating functions F_{nc} where $\lambda+\nu=c$. When $K_1=\cdots=K_n=K$ and $A_{n\nu}=1$ we get the case of Moessner with the result that $F_{n,1+K}=(1+nz)^K$ and consequently $A^{K}_{n,1}=n^K$ (which is Moessner's theorem). The generalization, as well as its proof, is itself a generalization of a result given by the present author for the case $K_1=\cdots=K_n=K$ [ibid. **1952**, 1–5; MR **14**, 846].

H. W. Brinkmann (Swarthmore, Pa.).

★**Gloden, A. Table des solutions de la congruence $x^4+1\equiv 0 \pmod{p}$ pour $350\,000<p<500\,000$.** Chez l'auteur, Luxembourg, 1946. 42 pp.

This is an extension of previous tables of A. Delfeld [Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. (N.S.) **16** (1946), 65–70; MR **8**, 564], and the author [Mathematica, Timişoara **21** (1945), 45–65; MR **7**, 145] and of the previous authors there cited. The present table gives for the 1446 primes p of the form $8n+1$ between the limits mentioned, the two absolutely least solutions of the congruence being considered. Nine examples of factorizations of n^4+1 are given in the introduction. Solutions were obtained from the quadratic partitions

$$p=a^2+b^2=c^2+2d^2$$

which are not given, but which can be recovered from the table by the use of continued fractions. *D. H. Lehmer.*

Rusu, Eugen. Des nombres représentables par la forme $F=a^2+kb^2$, $(a, b)=1$. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 273–286. (Romanian. Russian and French summaries)

Let p be any possible prime divisor of the quadratic form $F=a^2+kb^2$, $(a, b)=1$. In general, p itself need not be representable by F (e.g., for $k=11$, $p=5$ is a divisor of $2^2+11\times 1^2$, but $5\neq a^2+11b^2$ for any rational integral a,b). For specific numerical values of k, the author proves that either p, or some specified multiple of p is representable by F. The following is a characteristic result: For $k=47$, any prime divisor of a quadratic form $F=a^2+47b^2$ is such that either p, or $3p$, or $9p$ are representable by F, the alternatives being mutually exclusive. Hence, all prime divisors of this form F are divided into three classes. The product of two primes belonging to the same class is representable by F, the product of primes belonging to different classes is not. If $3p=F$, then also $16p=F$ and conversely; if $9p=F$, then also $8p=F$ and conversely. If p is any divisor of this F, then p^5 is representable by F. Twelve other results, of the same kind are proven. The proofs use elementary divisibility arguments and quadratic residues. *E. Grosswald.*

Scholz, B. Bemerkung zu einem Beweis von Wieferich. Jber. Deutsch. Math. Verein. **58** (1955), Abt. 1, 45–48.

Wieferich [Math. Ann. **66** (1909), 95–101] proved that every natural integer S is the sum of 9 or fewer positive cubes, with 23 and 239 the only known numbers actually requiring 9. Before Wieferich, 13 cubes were known to be sufficient (Maillet, Fleck). Wieferich left a minor error in his proof (affecting the range $7.4\times 5^{12}<S\leqq 7.4\times 5^{15}$). The proof was completed by Bachmann [Niedere Zahlentheorie, T. II, Teubner, Leipzig, 1910] (with a new error) and Kempner [Math. Ann. **72** (1912), 387–399].

The present paper gives in four pages a sensible simplification of Wieferich's proof, which covers also the exceptional range. *A. J. Kempner* (Boulder, Colo.).

Cellitti, Carlo. Sopra una proprietà delle forme quadratiche binarie primitive di determinante $D\equiv 1 \pmod 4$. Boll. Un. Mat. Ital. (3) **10** (1955), 527–530.

The author proves the following results (which are essentially a part of one of the standard proofs of the relation between the class-numbers of properly primitive and improperly primitive classical integral binary quadratic forms of the same discriminant, and which is proved in the usual way): (I) Every class of properly primitive forms $ax^2+2bxy+cy^2$ of discriminant $D=b^2-ac\equiv 1$ (4) contains a form for which $a\equiv 0$ (4), $b\equiv c\equiv 1$ (2). (II) If $a_ix^2+2b_ixy+c_iy^2$ $(1\leqq i\leqq h)$ are a set of representatives of this type, one from each class then $\frac{1}{2}a_ix^2+2b_ixy+2c_iy^2$ are a complete set of representatives of the classes with respect to the modular group $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, $\beta\equiv 0$ (2) of improperly primitive forms having the last coefficient $\equiv 2$ (4). *J. W. S. Cassels* (Cambridge, England).

Vrănceanu, G. On a theorem equivalent to Fermat's theorem. Gaz. Mat. Fiz. Ser. A. **8** (1956), 23–24. (Romanian)

Sastry, S. On problems allied to the "easier Waring problem". J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 87–89.

The author considers the form $u^4-v^4-r^3+s^3$ and shows that every odd number is of this form, even with $u=v+2$, and s restricted to 1, 3, 5, 7. In fact

$$8x+1=(x+1)^4-(x-1)^4-(2x)^3+1^3,$$
$$8x+3=(x-2)^4-(x-4)^4-(2x-6)^3+3^3,$$
$$8x+5=(x-14)^4-(x-16)^4-(2x-30)^3+5^3,$$
$$8x+7=(x-41)^4-(x-43)^4-(2x-84)^3+7^3.$$

There are a few comments about the form $u^4+v^4-r^4-s^4$ and examples of small combinations of fifth powers. *D. H. Lehmer* (Berkeley, Calif.).

Kanold, Hans-Joachim. Über zahlentheoretische Funktionen. J. Reine Angew. Math. **195** (1955), 180–191 (1956).

Let $f(n)$ be a mapping of the natural numbers into themselves. Let F denote the set of integers $\{f(n)\}$, and $d(F)$ the asymptotic density of F. Then $d(F)=0$ if $\lim f(n)/n=\infty$. Let $N_s=N_s(f)$ denote the set of all integers n to which there correspond distinct integers $n_1, n_2, \cdots, n_s$ such that $f(n_1)=f(n_2)=\cdots=f(n_s)=f(n)$. If $\lim f(n)/n=0$, then $d(N_s)=1$. If $d(F)=0$ and $\sum_{n\leqq m} f(n)/n=O(m)$, then $d(N_s)=1$.

Next let $f(n)$ be multiplicative, so that $f(n_1n_2)=f(n_1)f(n_2)$ if $(n_1, n_2)=1$, and let there exist integers $a>1$ and k such that a divides $f(p)$ for all $p>k$. Then $d(F)=0$ provided $\liminf f(n)/n>0$. Let L denote the set of integers n such that $w|f(n)=vn$, where v and w are any integers with w, however, bounded above. Then $d(L)=0$ if $\sum_{n\leqq m} f(n)/n=O(m)$.

Let N be the set of all non-square-free integers n having a bounded number of prime factors. For any given integer u define the set M of all $m=f(n)-un$. Then $d(M)=0$ if $f(n)\geqq n(up+1)/p$ where p is the smallest prime factor of n. (The equation $m=f(n)-un$ is suggested by amicable numbers, which satisfy such a relation with f the sum of divisors function, and $u=1$.)

There are further results along the same lines as the sample propositions given above. Many of the proofs are based on previous results of the author [Math. Z. **61** (1954), 180–185; J. Reine Angew. Math. **193** (1954), 250–252; MR **16**, 337; **16**, 569]. *I. Niven.*

Lambek, J.; and Moser, L. Rational analogues of the logarithm function. Math. Gaz. **40** (1956), 5–7.

The authors explore the use of the rational function

$$l_r(n)=h(rn)-h(r),$$

where $h(m)=\sum_{k=1}^{m} k^{-1}$ is the harmonic series as far as m terms, as a substitute for the transcendental function $\log x$ in making elementary proofs about the distribution of primes. Theorems like

$$|l_r(xy)-l_r(x)-l_r(y)|<2r^{-1}$$

are proved by elementary means for rational numbers x, y.

D. H. Lehmer (Berkeley, Calif.).

Sastry, S. On Prouhet-Lehmer problem. II. J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 90–92.

Let $P(k, s)$ denote the least number j such that there exist s different sets of j positive integers each having equal sums of nth powers for $n=1, 2, \cdots, k$. It is conjectured by Wright [Bull. Amer. Math. Soc. **54** (1948), 755–757; MR **10**, 101, 855] that $P(k, s)=k+1$ and this has been proved for $k<6$. In the author's previous paper he proved that $P(6, s)\leqq 5s+1$ [J. Sci. Res. Benares Hindu Univ. **1** (1950–1951), 1–4; MR **13**, 535]. In this paper the result is improved to $P(6, s)\leqq 4s$. Wright has proved that $P(6, s)\leqq 22$.

D. H. Lehmer (Berkeley, Calif.).

Analytic Number Theory

Cheo, Peh-Hsuin; and Yien, Sze-Chien. A problem on the k-adic representation of positive integers. Acta Math. Sinica **5** (1955), 433–438. (Chinese. English summary)

Let $x=\sum_{r=1}^{t} a_r k^{n_r}$, where $n_1>n_2>\cdots>n_t>0$, $1\leqq a_r\leqq k-1$, be the representation of the integer $x\geqq 1$ to the fixed integral basis $k\geqq 2$. Put $\alpha(x)=\sum_{r=1}^{t} a_r$, $A(x)=\sum_{y\leqq x}\alpha(y)$, and denote for fixed integral $m\geqq 1$ by $B_m(x)$ the number of integers $y\leqq x$ satisfying $\alpha(y)=m$. In the identity

$$A(x)=\frac{n_1(k-1)}{2}\sum_{r=1}^{t} a_r k^{n_r}-\frac{k-1}{2}\sum_{r=1}^{t}(n_1-n_r)a_r k^{n_r}$$
$$+\tfrac{1}{2}\sum_{r=1}^{t} a_r(a_r-1)k^{n_r}+\sum_{r=1}^{t} a_r+\sum_{r=1}^{t}\Big(\sum_{\sigma=1}^{r-1} a_\sigma\Big)a_r k^{n_r}$$

the first sum equals $\frac{1}{2}(k-1)[\log x/\log k]x$, while the four other sums are $O(x)$; hence

$$A(x)=\frac{k-1}{2}\,\frac{x\log x}{\log k}+O(x). \tag{1}$$

Here the error term is best possible, as is seen on taking $x=(k-1)k^n$. The author further shows that

$$B_m(x)\sim\frac{1}{m!}\Big(\frac{\log x}{\log k}\Big)^m \text{ as } x\to\infty.$$

[For the case $k=2$ of (1) see Bellman and Shapiro, Ann. of Math. (2) **49** (1948), 333–340; MR **9**, 414.]

K. Mahler (Manchester).

Hornfeck, Bernhard. Basen mit paarweise teilerfremden Elementen. Arch. Math. **7** (1956), 49–51.

Schnirelmann proved that the set $\mathfrak{P}$ of primes, with the addition of 0 and 1 form an additive basis of finite order for the natural numbers. The author proves a theorem from which it follows that this result remains true when $\mathfrak{P}$ is replaced by a set of integers $\{0, a_1=1, a_2, a_3, \cdots\}$ of similar density having the property that $(a_i, a_j)=1$ for $i\neq j$.

R. A. Rankin (Glasgow).

Golomb, Solomon W. Sets of primes with intermediate density. Math. Scand. **3** (1955), 264–274 (1956).

An infinite set of primes is said to have intermediate density if it has a distribution which does not exceed $Ax/\log x$ for any constant $A>0$. In this article the author exhibits a large family of such sets of primes. Two interesting examples are the following. (1) Let G_1 be the regular semi-group obtained by going consecutively through the odd primes, and allowing each prime as a generator of G_1 so long as it does not conflict with any previous generator. Then the set of primes which generates G_1 has intermediate density. (2) Let P_n be the largest prime divisor of the Fermat number $F_n=2^{2^n}+1$. Then the set $\{P_n\}$ has intermediate density, and the series $\sum 1/P_n$ converges.

A. L. Whiteman.

Bochner, S. Gamma factors in functional equations. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 86–89.

The gamma factors in the functional equation for a zeta function usually arise from an expression of the form

$$\eta(x)\mu(x)^s\int_P e^{-(x,t)}R(t)^s d\Omega(t). \tag{1}$$

Here P is an open set in real Euclidean k-space, Ω is a positive set function, $R(t)$ is a polynomial in $t_1, \cdots, t_k$, $R(t)$ is positive on P; (x, t) denotes the inner product $x_1t_1+\cdots+x_kt_k$; $\eta(x)$ and $\mu(x)$ are positive functions of $x_1, \cdots, x_k$, defined and infinitely differentiable over an open set G. It is assumed that the integral converges absolutely, if $x\in G$, $\operatorname{Re} s>\sigma_0$. Assuming, moreover, that the value $\Delta(s)$ of (1) does not depend on x (if $x\in G$), the author shows that $\Delta(s)$ has the form $Ae^{as}\prod_{m=1}^{n}\Gamma(s-a_m)$, where $A>0$, a real, $a_1, \cdots, a_n$ complex.

N. G. de Bruijn (Amsterdam).

See also: Brodskiĭ, p. 720; Urazbaev, p. 829; Chowla and Mientka, p. 829; Davenport and Roth, p. 829.

Algebraic Number Theory

Carlitz, L.; and Hodges, John H. Distribution of bordered symmetric, skew and hermitian matrices in a finite field. J. Reine Angew Math. **195** (1955), 192–201 (1956).

Let A be a non-singular symmetric matrix of order m over a finite field $\mathrm{GF}(q)$, q being odd. Making use of results established in previous papers [Carlitz, Duke Math. J. **21** (1954), 123–137; Arch. Math. **5** (1954), 19–31; Math. Nachr. **11** (1954), 135–142; MR **15**, 604, 777, 778] the authors first derive formulas for the number of $m\times t$ matrices U with elements in $\mathrm{GF}(q)$ such that the matrix

$$M=\begin{vmatrix} A & U \\ U' & 0 \end{vmatrix}, \tag{*}$$

where U' denotes the transpose of U, has rank $m+r$, $0\leqq r\leqq m$, t and invariant λ. They then deal with the more general case of (*) in which the rank of A does not exceed m. Analogous formulas for skew-symmetric matrices and hermitian matrices over $\mathrm{GF}(q^2)$ are also obtained.

A. L. Whiteman (Los Angeles, Calif.).

Hunter, John. A generalization of the inequality of the arithmetic-geometric means. Proc. Glasgow Math. Assoc. **2** (1956), 149–158.

Continuing improvements of estimates like Schur's and Siegel's [Siegel, Ann. of Math. (2) **46** (1945), 302–312; MR **6**, 257] on symmetric functions of the conjugates x_i $(1\leqq i\leqq n)$, of a totally positive algebraic integer, the author considers lower bounds for $\sum x_i^2/n$ $(>4.654\cdots-\varepsilon)$, for

$\sum_{i<j}(x_i-x_j)^2/n^2$ ($>1.6487\cdots-\varepsilon$), and for $\sum_{i<j}x_ix_j/n^2$ ($>1.7336\cdots-\varepsilon$) with n large. *Harvey Cohn.*

Newman, M.; and Taussky, Olga. On a generalization of the normal basis in abelian algebraic number fields. Comm. Pure Appl. Math. **9** (1956), 85–91.

Let K be a finite normal extension of the rational field r. Let $\{S_k\}$ ($k=1, \cdots, n$), be the elements of the Galois group of K/r. If $\{\alpha_i\}$ are a basis for the integers of K then the authors call $D=(\alpha_i^{S_k})$ a discriminant matrix for K/r. A basis $\{\alpha_i\}$ is a normal basis if it is of form $\{\alpha^{S_i}\}$ for some $\alpha \in K$. A normal basis for the integers exists for some but not all K/r. A matrix is called normal if it commutes with its conjugate transpose. By matrix calculation the authors prove: 1. If K/r is abelian, then any discriminant matrix derived from a normal basis is normal. 2. If K/r is cyclic, the normal bases, or those derived from them by permutation and multiplication by -1, are the only ones for which D is normal. 3. If K/r is cyclic and has a normal basis for the integers, then this normal basis is unique (except for the changes mentioned in 2.) when the degree is 2, 3, 4, or 6, and non-unique in all other cases.

G. Whaples (Princeton, N.J.).

Urazbaev, B. M. An asymptotic formula for growth of the number of Abelian fields of type $(l, l, \cdots, l)$. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 659–661. (Russian)

A proof is sketched that the number of abelian fields of discriminant $\leqq x$ whose group is the direct product of k cyclic groups of order l (a prime) is given by

$$xf(\log x)+O(x^{1-(l^k-1)/l^{k-1}+\varepsilon}),$$

where $f(y)$ is a polynomial of degree $(l^k-1)/(l-1)$ whose coefficients depend only on l and k. The method is similar to one used in a slightly different context [B. M. Urazbaev, same Dokl. (N.S.) **95** (1954) 1145–1147; MR **15**, 937].

J. W. S. Cassels (Cambridge, England).

See also: van der Blij, p. 820.

Geometry of Numbers, Diophantine Approximation

Chowla, S.; and Mientka, W. E. The number of lattice points in an n-dimensional tetrahedron. Proc. Amer. Math. Soc. **7** (1956), 51–53.

Let $a_1, \cdots, a_n$ and ξ be positive integers. By a simple analytic argument it is shown that the number of solutions in integers $x_i \geqq 0$ of $\sum a_ix_i=\xi\prod a_i$ and of $\sum a_ix_i \leqq \xi\prod a_i$ are polynomials in the a_i, ξ with rational coefficients.

J. W. S. Cassels (Cambridge, England).

Davenport, H.; and Swinnerton-Dyer, H. P. F. Products of inhomogeneous linear forms. Proc. London Math. Soc. (3) **5** (1955), 474–499.

Suppose $L_k=\sum_{j=1}^{n}a_{kj}x_j$, where $\Delta=\det(a_{kj})\neq 0$, a_{kj} is real if $k=1, \cdots, r$ and $j=1, \cdots, n$, the remaining coefficients are complex and $a_{k+s,j}=\bar{a}_{kj}$ if $k=r+1, \cdots, r+s$ and $j=1, \cdots, n$. Let $r+2s=n$, $P(x_1, \cdots, x_n)=L_1L_2\cdots L_n$, $M_H=\min|P(x_1, \cdots, x_n)|$ taken over integers $x_1, \cdots, x_n$, not all zero, $M_I(x')=\min|P(x_1+x_1', \cdots, x_n+x_n')|$ taken over all integers $x_1, \cdots, x_n$, where $x_1', \cdots, x_n'$ are real numbers, and $M_I=\max M_I(x')$ taken over all sets of real numbers $x_1', \cdots, x_n'$.

If $r>0$ and $s>1$, the authors prove that the inequality

$$M_H^{\lambda}M_I^{1-\lambda}<C|\Delta| \tag{1}$$

holds with

$$\lambda=\max\left(\frac{s-1}{n-1}, \frac{s}{2n-2s}\right).$$

This is an improvement on earlier results; for instance, Davenport [Quart. J. Math. Oxford Ser. (2) **3** (1952), 32–41; MR **13**, 918] first proved that (1) holds with $\lambda=s/n$. More refined results are also proved in the cases when $s=2$ and $r=0$ or 1. *H. S. A. Potter* (Aberdeen).

Davenport, H.; and Roth, K. F. The solubility of certain Diophantine inequalities. Mathematika **2** (1955), 81–96.

Let $\lambda_1, \cdots, \lambda_s$ be real numbers, not all of the same sign (say $\lambda_1\lambda_2<0$) and not all rational multiples of a single irrational number. Let $\varkappa$ be an arbitrary real number and let ε be positive. Improving a result of Davenport and Heilbronn [J. London Math. Soc. **21** (1946), 185–193; MR **8**, 565], it is shown that there is an absolute constant C such that if $k \geqq 12$ and $s>Ck\log k$, or if $k=3$ and $s=8$, the inequality

$$|\lambda_1x_1^k+\cdots+\lambda_sx_s^k+\varkappa|<\varepsilon$$

has infinitely many solutions in positive integers $x_1, \cdots, x_s$. *W. J. LeVeque* (Ann Arbor, Mich.).

Cugiani, Marco. Sopra una questione di approssimazione diofantea non lineare. Boll. Un. Mat. Ital. (3) **10** (1955), 489–497.

The author considers the set $I(\alpha)$ of points $p^2-q^2\alpha$, where α is a positive real number and p, q are integers by considering the continued fraction development of $\alpha^{\frac{1}{2}}$. He proves the following results. (I) For almost all α the point 0 is a point of accumulation of $I(\alpha)$ both on the left and right. Then $I(\alpha)$ is dense on the real axis. (II) There exist α for which 0 is an isolated point of $I(\alpha)$ and also α for which 0 is a point of accumulation only on left (or right). (III) $I(\alpha)$ has points of accumulation γ, δ in $0<\gamma\leqq 2\alpha^{\frac{1}{2}}+1$ and $0>\delta>-2\alpha^{\frac{1}{2}}-1$. The proofs are quite straightforward on noting that if $\beta \in I(\alpha)$ then $n^2\beta \in I(\alpha)$ for all integers n. *J. W. S. Cassels.*

ANALYSIS

Matorin, A. P. On inequalities between the maxima of the absolute values of a function and its derivatives on a half-line. Ukrain. Mat. Ž. **7** (1955), 262–266. (Russian)

Let M_k denote the least upper bound of $|f^{(k)}(x)|$. For a given interval there is an inequality $M_k \leqq C_{nk}M_0^{(n-k)/n}M_n^{k/n}$. When the interval is $(-\infty, \infty)$ the exact C_{nk} was obtained by Kolmogoroff [Moskov. Gos. Univ. Uč. Zap. **30**, Mat. **3** (1939), 3-16=Amer. Math. Soc. Transl. no. **4** (1949); MR **1**, 298; **11**, 86]. For finite or semi-infinite intervals the best results so far were given by Gorny [Acta Math. **71** (1939), 317–358; MR **1**, 137, 400] and H. Cartan [Sur les classes de fonctions définies par des inégalités portant sur leur dérivées successives, Hermann, Paris, 1940; MR **3**, 292]. The author obtains the following improved value for C_{nk} for the interval $(0, \infty)$:

$$\frac{n^2(n^2-1^2)(n^2-2^2)\cdots[n^2-(k-1)^2][(2n-1)!!]^{k/n}}{\{n^2(n^2-1^2)\cdots[n^2-(n-1)^2]\}^{k/n}(2k-1)!!}.$$

This is not shown to be the best possible value, although it is for $n=2$ and also $n=3$, $k=1$; in the second case an extremal function is given by $T_3(x)$ for $-1\leq x\leq 1/2$, $-T_3(x-1)$ for $1/2\leq x\leq 3/2$, $T_3(x-2)$ for $3/2\leq x\leq 5/2$, etc., where $T_n(x)$ is the nth Čebyšev polynomial $\cos(n\cos^{-1}x)$. Numerically, $C_{31}=3\cdot 9^{1/3}/2=3\cdot 12$, where Cartan's value was $6e^2>44$ (and Gorny's was twice this). The author's C_{nk} is the value for which

$$M_k=C_{nk}M_0^{(n-k)/n}M_n^{k/n}$$

for $T_n(x)$ on $[-1, 1]$. His proof depends on comparing $f(x+1/b)$ with $\phi_n(x)=aT_n(bx)$ on $(-1/b, 1/b)$, where a and b are chosen so that $M_k(\phi_n)=M_k(f)$ and $M_n(\phi_n)=M_n(f)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Nikol'skiĭ, S. M. On an inequality for periodic functions. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 219–222. (Russian)

The author proves an inequality for functions of n real variables; for simplicity of notation we state it here for $n=2$. The class $H_p^{(q,r)}(M_1, M_2)$ consists of functions $f(x, y)$ of period 2π in each variable with $q=s+\alpha$, $r=t+\beta$, belonging to L^p qua function of two variables, such that

$$\|\Delta_h f_x^{(s)}(x, y)\|_p\leq M_1h^\alpha \text{ and } \|\Delta_h f_y^{(t)}(x, y)\|_p\leq M_2h^\beta$$

if $0<\alpha<1$ and $0<\beta<1$; if $\alpha=1$ or $\beta=1$, the corresponding Δ is replaced by Δ^2. Here $f_x^{(s)}$ means $\partial^s f/\partial x^s$, etc., and the difference is with respect to the same variable as the derivative. Then if f has mean value zero over a period, and $1\leq p\leq\infty$, the inequality states that $\|f\|_p\leq c(M_1+M_2)$, where c depends only on p, q and r. *R. P. Boas, Jr.*

Harrik, I. Yu. On an analogue of Markov's inequality. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 203–206. (Russian)

The author proves the following theorem: Let D be a bounded closed set in m-space, Γ its boundary, $\varphi(X)=\varphi(x_1, x_2, \cdots, x_m)$ a function defined on an open set containing D and satisfying: 1) $\varphi\in C^2$, 2) $\varphi=0$ on Γ, 3) $\varphi\neq 0$ off Γ, 4) grad $\varphi\neq 0$ on Γ. Then if $P(X)=P(x_1, x_2, \cdots, x_m)$ is a polynomial of degree not more than n in each variable, the following inequality relations obtain (A, A^*, A^{**} are independent of P and n): a) $\|P\|_{C(D)}\leq An^2\|\varphi P\|_{C(D)}$; b) $\|\partial(\varphi P)/\partial x_k\|_{C(D)}\leq A^*n^2\|\varphi P\|_{C(D)}$; c) $\|\varphi\partial P/\partial x_k\|_{C(D)}\leq A^{**}n^2\|\varphi P\|_{C(D)}$, where $\|f\|_{C(D)}=\sup\{|f(X)|\ :\ X\in D\}$. As in earlier writings [Mat. Sb. N.S. **37(79)** (1955), 353–354; MR **17**, 256] the author applies these results to the convergence rate estimation problem connected with the Ritz technique for solving self-adjoint partial differential equations of elliptic type with boundary conditions. *B. Gelbaum.*

Ionescu, D. V. Généralisation de la formule de Taylor. Gaz. Mat. Fiz. Ser. A. **7** (1955), 389–395. (Romanian. Russian and French summaries)

The author presents, as a generalization of the classical Taylor formula, the expression

$$y(x)=y_1(x)y(x_0)+\cdots+y_n(x)y^{(n-1)}(x_0)+\int_{x_0}^{x}K(x, s)F[y(s)]ds,$$

where, for a given differential equation

$$F(y)\equiv y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_n(x)y=0$$

with continuous coefficients $a_j(x)$ and for the Kronecker δ's, the functions $y_1(x), \cdots, y_n(x)$, $K(x, s)$ are integrals of $F=0$ which satisfy $y_i^{(k)}(x_0)=\delta_{ik}$ and $\partial^t K(x, s)/\partial x^t|_{x=s}=\delta_{t,n-1}$. *E. F. Beckenbach* (Los Angeles, Calif.).

Grimshaw, M. E. On Taylor's theorem. Proc. Cambridge Philos. Soc. **52** (1956), 376–378.

The author establishes, under rather weak conditions on the derivatives involved, a general formula for the remainder $R(n, a, b)$ in Taylor's theorem. Thus if $g(\xi)$ is a finite nonvanishing Lebesgue-integrable function in $a\leq x<b$, with $g(x)=(d/dx)\int_a^x g(\xi)d\xi$, and the integral

$$G(b)=\frac{1}{(n-1)!}\int_a^b(b-\xi)^{n-1}g(\xi)\,d\xi$$

is convergent for some positive integer n, then for any function $f(x)$ continuous in $a\leq x<b$ and such that $f'(x), \cdots, f^{(n-1)}(x)$ are continuous in $a\leq x<b$ and $f^{(n)}(x)$ exists, finite or infinite, in $a<x<b$, there exists a value ξ, $a<\xi<b$, such that

$$f(b)=\sum_{\nu=0}^{n-1}\frac{(b-a)^\nu}{\nu!}f^{(\nu)}(a)+G(b)\frac{f^{(n)}(\xi)}{g(\xi)}.$$

The choice $g(\xi)=(b-\xi)^{p-n}$ for $p>0$ gives the Schlömilch form of the remainder, including the Lagrange form ($p=n$) and the Cauchy form ($p=1$), and a simple deduction yields the integral form. *E. F. Beckenbach.*

Gyires, Béla. Über die Grenzwerte von Matrizen, die den Cauchyschen Funktionalgleichung genügen. Acta Univ. Debrecen **1** (1954), 136–144; additamentum ad **1** (1955), 19. (Hungarian. German summary)

In dieser Arbeit beschäftigt sich Verf. mit Funktionalmatrizen der Form

$$A(t)=\begin{pmatrix}a_{11}(t)\cdots a_{1n}(t)\\ \cdot\quad\cdots\quad\cdot\\ a_{n1}(t)\cdots a_{nn}(t)\end{pmatrix} \tag{1}$$

deren Elemente auf der nichtnegativen reellen Geraden definierte, im Nullpunkt differenzierbare reelle stetige Funktionen sind, welche den Bedingungen $A(t_1+t_2)=A(t_1)A(t_2)$ und $A(0)=(1)_n$ genügen, und keine von 1 verschiedene Eigenwerte vom Modul 1 haben. Es wird gezeigt, dass die Grenzmatrix $\lim_{t\to\infty}A(t)=A$ dann und nur dann existiert, falls die zum Eigenwert $A(1)$ gehörigen Elementarteiler der Matrix $\lambda=1$ linear sind. Die Grenzmatrix wird in diesem Falle durch $A=P((1)_r, (0)_{n-r})P^{-1}$ gegeben, wo P eine der Matrizen ist, welche (1) oder $B=[dA(t)/dt]_{t=0}$ in die Jordansche Normalform überführen. Das gewonnene Resultat findet auf stochastische Funktionalmatrizen Anwendung.

Author's summary.

Natanson, I. P. An addition to Hausdorff's theorems on moment sequences. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 191–194. (Russian)

Suppose that a sequence $\mu_m, \mu_{m+1}, \cdots$ satisfies as much as it can of a set of conditions for a Hausdorff moment sequence, for example

$$\sum_{k=m}^{n}\binom{n}{k}|\Delta^{n-k}\mu_k|<H\quad(n=m, m+1, \cdots).$$

Then it is part of a Hausdorff moment sequence $\mu_0, \mu_1, \cdots$; and (if $m>0$) μ_0 can be assigned arbitrarily and (if $m>1$) $\mu_1, \cdots, \mu_{m-1}$ are then determined.

R. P. Boas, Jr. (Evanston, Ill.).

Farinha, João. On a case of convergence of continued fractions with complex elements. Gaz. Mat., Lisboa **12** (1951), no. 50, 81. (Portuguese)

The object of this note is the proof of the following sufficient condition for the convergence of a certain class of continued fractions with complex elements: Let $f_1(z)$, $f_2(z)$, $\cdots$, $f_n(z)$, $\cdots$ be functions of a complex variable which satisfy in a domain D the following conditions: $|f_1|\geqq 1+a_1$, $|f_n|\geqq 1+a_n+1/(1+a_{n-1})$, $n=2, 3, \cdots$, $\liminf (a_n)\geqq a>0$. Then the continued fraction

$$\frac{1}{f_1}+\frac{1}{f_2}+\cdots+\frac{1}{f_n}+\cdots$$

is uniformly convergent in D, and all the approximants lie in $|z|\leq(a+1)/a(a+2)$. *E. Frank* (Chicago, Ill.).

★**Severi, Francesco; e Scorza Dragoni, Giuseppe. Lezioni di analisi. Vol. II. Serie di funzioni, applicazioni geometriche, integrali rettilinei, funzioni di più variabili, derivazione e integrazione ad esse inerenti. 4a ed.** Cesare Zuffi, Bologna, 1955. vii+398 pp. 4500 Lire.

For a review of the first edition (1942) see MR **10**, 238. This edition shows few changes.

See also: Hunter, p. 828.

Theory of Sets, Functions of Real Variables

Fodor, G. Generalization of a theorem of Alexandroff and Urysohn. Acta Sci. Math. Szeged **16** (1955), 204–206.

Let λ be a limit ordinal number that is not cofinal with ω, and M be a subset of $W(\lambda)$ such that $W(\lambda)-M$ contains no closed subset with which $W(\lambda)$ is cofinal. Suppose that to every $\alpha\in M-\{0\}$ there corresponds an ordinal number $\varphi(\alpha)<\alpha$. Then there exists an ordinal number $\beta<\lambda$ and a set $N\subset M$, such that $W(\lambda)$ is cofinal with N and, for every $\alpha\in N$, $\varphi(\alpha)\leqq\beta$. This generalizes a result of Bachmann [Transfinite Zahlen, Springer, Berlin, 1955, p. 43; MR **17**, 134]. *F. Bagemihl* (Notre Dame, Ind.).

Tóth, E. Sur le puissance dénombrable des polynômes à coefficients entières. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 55–59. (Romanian. Russian and French summaries)

Let $p_0=2, p_1=3, p_2=5, \cdots, p_n, \cdots$ be the sequence of prime numbers. A one-to-one correspondence is established between the set of positive rational numbers and the set of polynomials in x with integral coefficients, by associating

$$(p_{m_1}^{a_1}\cdots p_{m_j}^{a_j})/(p_{n_1}^{b_1}\cdots p_{n_k}^{b_k}) \text{ with}$$
$$a_1x^{m_1}+\cdots+a_jx^{m_j}-b_1x^{n_1}-\cdots-b_kx^{n_k},$$

where $a_1, \cdots, a_j, b_1, \cdots, b_k$ denote natural numbers. *F. Bagemihl* (Notre Dame, Ind.).

Hayes, C. A., Jr. The Heine-Borel theorem. Amer. Math. Monthly **63** (1956), 180–182.

Locher-Ernst, L. Wie man aus einer Kugel zwei zu ihr kongruente Kugeln herstellen kann. Elem. Math. **11** (1956), 25–35.

Expository article, designed to serve as an introduction and companion to R. M. Robinson's treatment of the subject [Fund. Math. **34** (1947), 246–260; MR **10**, 106]. *P. R. Halmos* (Chicago, Ill.).

Stoilov, S. Sur quelques questions concernant les fondements de l'analyse classique. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 20–24. (Romanian. Russian and French summaries)

The real numbers are defined essentially by Cauchy sequences of finite decimal rationals. *E. Grosswald.*

Holladay, John; and Sobczyk, Andrew. An equivalent condition for uniform convergence. Amer. Math. Monthly **63** (1956), 31–33.

The authors prove the following theorem: Let S be a compact space, and R a space with a metric $d[.,.]$. Let $\{f_n\}$ be a sequence of continuous mappings on S to R, which converges pointwise to a mapping g. Then $\{f_n\}$ converges uniformly if and only if (1) g is continuous and (2) for arbitrary $\varepsilon>0$ there exists an integer $m\geqq 0$ and a $\delta>0$, such that whenever $n>m$ and $d[f_i(s), g(s)]<\delta$, then $d[f_{i+n}(s), g(s)]<\varepsilon$. The authors remark that if the f_n's are iterations of a single mapping, m may be taken to be zero. The theorem is also generalized to Moore-Smith sequences. *A. Rosenthal* (Lafayette, Ind.).

Tarnawski, E. On the spaces of functions satisfying Hölder's condition. Fund. Math. **42** (1955), 207–214.

Let $w(h)$ and $w_1(h)$ be positive and non-decreasing for $h>0$ and tend to 0 as $h\to 0$. Let H_w be the set of all functions f with period 1 which admit $w(h)$ as modulus of continuity. A distance is defined in H_w: $d(f, g)=\sup|f(x)-g(x)|$. This makes H_w into a complete metric space. Let R be the set of all $f\in H_w$ such that

$$\limsup_{h\to 0}|f(x+h)-f(x)|/w_1(|h|)=\infty$$

for every x. Under the assumption that $\lim_{h\to 0} h/w(h)=0$ it is proved that R is either empty or residual in H_w. More detailed results show under what conditions these two cases occur. *W. Rudin* (Rochester, N.Y.).

Campos Ferreira, Jaime. Sur la notion d'ordre infinitesimal. Portugal. Math. **14** (1955), 43–62.

Let S be any set and let $\mathfrak{F}$ be a filter on S; designate by $\Phi_{\mathfrak{F}}$ the family of all non-negative functions whose domains belong to $\mathfrak{F}$, and let φ, ψ be two functions of $\Phi_{\mathfrak{F}}$. The author calls φ "finer" than ψ (with respect to $\mathfrak{F}$), denoted by $\varphi\sqsupset\psi$ ($\mathfrak{F}$), if there exists a set $F\in\mathfrak{F}$ and a number $k>0$ such that for every $x\in F$ one has $\varphi(x)\leqq k\psi(x)$. If $\varphi\sqsupset\psi$ and $\psi\sqsupset\varphi$, then φ and ψ are said to have "the same infinitesimal order", designated by $\varphi\square\psi$ ($\mathfrak{F}$). This is the case if and only if there exists a set $F\in\mathfrak{F}$ and a pair of positive numbers a, b such that for every $x\in F$ one has $a\varphi(x)\leqq\psi(x)\leqq b\varphi(x)$. For every $\varphi\in\Phi_{\mathfrak{F}}$, the equivalence class containing φ is called "the infinitesimal order" of φ (with respect to $\mathfrak{F}$) and denoted by ord φ. In the set $\mathfrak{A}$ of the infinitesimal orders the operations of addition and of multiplication with non-negative numbers α are defined by ord φ+ord ψ =ord $(\varphi\psi)$ and $\alpha\cdot$ord φ=ord (φ^α), respectively. Then $\mathfrak{A}$ is an abelian semigroup. If $\mathbf{a}$=ord φ, $\mathbf{b}$=ord ψ and $\varphi\sqsupset\psi$, then also the notation $\mathbf{a}\sqsupset\mathbf{b}$ is used. If $\mathbf{a}\sqsupset\mathbf{b}$, but $\mathbf{a}\neq\mathbf{b}$, then the author calls $\mathbf{a}$ "strictly finer" than $\mathbf{b}$ and writes $\mathbf{a}>\mathbf{b}$. On the other hand, he calls $\mathbf{a}$ "greater" than $\mathbf{b}$ and writes $\mathbf{a}>\mathbf{b}$ if there exists a function $\omega\in\Phi_{\mathfrak{F}}$ and a set $F\in\mathfrak{F}$ such that $\varphi(x)=\omega(x)\cdot\psi(x)$ for every $x\in F$ and $\lim_{\mathfrak{F}}\omega(x)=0$. Relations between these notions are discussed.

Then the author considers the subset $\Phi_{\mathfrak{F}}^*$ of $\Phi_{\mathfrak{F}}$ consisting of all functions $\varphi\in\Phi_{\mathfrak{F}}$ for which there exists an

$F \in \mathfrak{F}$ such that $\varphi(x)>0$ for every $x \in F$. The subset $\mathfrak{A}^*$ of $\mathfrak{A}$ which corresponds to $\Phi_{\mathfrak{F}}^*$ is a Riesz space with respect to the relation $\sqsupset$. Supposing that the filter $\mathfrak{F}$ admits a denumerable basis $\{B_n\}$ such that $\bigcap_{n=1}^{\infty} B_n=\emptyset$, the author proves the possibility of majorizing (minorizing) every sequence which is increasing (decreasing) with respect to the above order relations and he obtains also some results concerning upper (lower) bounds of such sequences.

Finally, the author generalizes his results for the case of functions whose values belong to a normed space.

A. Rosenthal (Lafayette, Ind.).

Baiada, E.; e Vinti, C. Un'applicazione della definizione di integrale per stabilire un passaggio al limite sotto il segno. Boll. Un. Mat. Ital. (3) **10** (1955), 460–464.

It is proved that each of the expressions

$$\int_a^b dy \int_{y-d}^{y+d} M(t)\,dt \text{ and } \int_a^b K(y)dy \int_{y-d}^{y+d} M(t)\,dt$$

behaves as an infinitesimal of first order in d, provided that in the first case $M(t)$ be non-negative and summable, in the second case $K(y)$ and $M(t)$ be square-summable in the interval $(a-2d, b+2d)$. *A. Rosenthal.*

See also: Denjoy, p. 832; Yûjôbô, p. 836.

Theory of Measure and Integration

Kestelman, H. Riemann equivalence of functions. Mathematika **2** (1955), 97–104.

Two finite real functions $f(x)$ and $g(x)$, defined for $-\infty<x<\infty$, are said to be Riemann equivalent if $|f(x)-g(x)|$ has a zero Riemann integral over every finite interval; we then write $f\sim g$. N. G. de Bruijn conjectured that if $f(x+h)\sim f(x)$ for every real number h, then $f\sim c$ where c is a constant; this was proved by P. Erdös [Nederl. Akad. Wetensch. Proc. Ser. A. **55**=Indag. Math. **14** (1952), 142–144; MR **13**, 830]. We associate with an arbitrary function f the additive group $\mathcal{P}(f)$ of all numbers h which make $f(x)\sim f(x+h)$. We show that in general $\mathcal{P}(f)$ is a set of pseudo-periods of f, in the sense that f is Riemann equivalent to a function which has $\mathcal{P}(f)$ for its group of periods. This happens whenever $\mathcal{P}(f)$ has positive exterior measure or is of the second category; it happens trivially when $\mathcal{P}(f)$ is discrete, but it need not happen when $\mathcal{P}(f)$ is countable and dense. The functions f which have interest for this discussion are all non-measurable in the Lebesgue sense; it is easily shown that a measurable function f for which $\mathcal{P}(f)$ is not discrete is necessarily equal to a constant apart from a set of measure zero. (From the author's introduction.) *R. P. Boas, Jr.*

Kovan'ko, A. S. The Stieltjes-Lebesgue integral of functions of two independent variables with two additional functions. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. **1** (1941), no. 1, 10–26. (Russian)

The author defines, in a domain G of the (u, v)-plane, Lebesgue integration with respect to a pair of functions of the form $(\phi_1-\phi_2, \psi_1-\psi_2)$, where, for $i=1, 2$ and $k=1, 2$, each (ϕ_i, ψ_k) defines an orientation-preserving homeomorphism of G onto a corresponding plane domain. His definition, which is shown to depend only on the original pair, not on the (ϕ_i, ψ_k), amounts to that of integration with respect to the appropriate combination of additive set-functions $\Phi_{ik}(E)$, where $\Phi_{ik}(E)$ coincides for closed sets E with the measure of the image of E under the homeomorphism defined by (ϕ_i, ψ_k). Surface area, for certain triples of functions, is also treated briefly. There are numerous minor misprints. *L. C. Young.*

Dubovickiĭ, A. Ya. On the definition of a linear integral for once differentiable functions and for functions satisfying a Lipschitz condition. Vologod. Ped. Inst. Uč. Zap. **11** (1953), 179–186. (Russian)

The author quotes an alternative definition, given by Kronrod [Uspehi Mat. Nauk (N.S.) **5** (1950), no. 1(35) 24–134; MR **11**, 648] for the notion of linear integral in the case of a continuously differentiable integrand; he remarks that this alternative definition remains valid for an integrand which is once differentiable or else Lipschitzian.

L. C. Young (Madison, Wis.).

Denjoy, Arnaud. Théorème de Vitali et intégration. C. R. Acad. Sci. Paris **240** (1955), 1385–1388.

This paper starts with some remarks about regularity, perfect regularity, and mean regularity of coverings. Then the author shows how the Borel measure in R_n can be mapped into the Borel measure in R_1 such that the value of the measure remains unchanged; he does this, for $n=2$, by means of a suitable Peano curve which covers the plane. He considers then also the case of more general measures defined for the Borel sets of R_2 and he discusses the transformation of Stieltjes integrals over bounded sets of R_2 into Lebesgue integrals over linear sets.

A. Rosenthal (Lafayette, Ind.).

Pauc, Christian; et Rutovitz, Denis. Essai d'une théorie de Ward-Denjoy pour fonctions de cellule. C. R. Acad. Sci. Paris **240** (1955), 1956–1958.

After having introduced numerous definitions and several axioms the authors greatly generalize the Ward-Denjoy theory, replacing the systems of intervals by systems of cells, deleting the Vitali conditions, and using Denjoy integrands defined by means of sequences of Radon-Nikodym integrands. *A. Rosenthal.*

Ionescu Tulcea, C. T. Sur l'intégration des fonctions d'ensemble. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 11–16. (Romanian. Russian and French summaries)

The author continues his study of general integration theory. Here the functions are set functions with values in an abelian topological semigroup. The resulting integral generalizes integrals considered by Kolmogoroff [Math. Ann. **103** (1930), 654–696], and the reviewer [Trans. Amer. Math. Soc. **52** (1942), 498–521; MR **4**, 162].

C. E. Rickart (New Haven, Conn.).

Ionescu Tulcea, C. T. L'intégrale dans les espaces ordonnés. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1953), no. 2, 9–14. (Romanian. Russian and French summaries)

The author considers an integration process with respect to a measure with values in an ordered vector space. In this setting he obtains a generalization of the Riesz representation theorem for linear functionals which is then applied to obtain spectral decomposition theorems for operators on a Hilbert space. *C. E. Rickart.*

Berstein, I. Sur les classes des fonctions équivalentes et sur les fonctions définies dans une algèbre de Boole. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Şci. Mat. Fiz. **7** (1955), 565–581. (Romanian. Russian and French summaries)

If f is a real-valued measurable function on a measure space with finite measure m, write $m_f(A, B)=m(f^{-1}(A)\cap B)$ whenever A is a Borel subset of the real line and B is a measurable subset of the space. The author gives an intrinsic characterization of functions such as m_f and proves that they uniquely determine the corresponding f. He also studies the way in which the convergence in measure of a sequence of functions f_n is reflected in the behavior of the corresponding set functions m_{f_n}. The set functions m_f indicate a way of defining the abstract Boolean algebraic analog (at least for measure algebras) of a measurable function on a measure space.

P. R. Halmos (Chicago, Ill.).

Civin, Paul. Some ergodic theorems involving two operators. Pacific J. Math. **5** (1955), 869–876.

Let u and t be two measure-preserving transformations acting on a measure space of elements s. Let f be an integrable function, and g a positive measurable function, defined on the measure space. Put $f_n(s)=\sum_{j=0}^{n} f(t^j s)$, $g_n(s)=\sum_{j=0}^{n} g(u^j s)$. Then, for almost all s, the quotient of $f_n(s)$ by $g_n(s)$ approaches a limit as $s\to\infty$. This generalizes an ergodic theorem of Hopf [Ergodentheorie, Springer, Berlin, 1937]. A corresponding generalization of the ergodic theorem of Hurewicz [Ann. of Math. (2) **45** (1944), 192-206; MR **5**, 148] is given. The proof is by reduction to the theorem of Hurewicz. The paper concludes with an application to the individual ergodic theory of Markov processes.

[Author's remark: In theorem 1 the result is established only for finite measure spaces unless both t and u are measure preserving.] *J. Schwartz* (New York, N.Y.).

Civin, Paul. Abstract Riemann sums. Pacific J. Math. **5** (1955), 861–868.

Let $T(\varepsilon)$ be a 1-parameter group of measure-preserving transformations on a measure space of elements s. Generalizing a theorem of Jessen, the author proves that for each integrable function f satisfying $f(s)=f(T(1)s)$ the sequence of sums $f_n(s)=2^{-n}\sum_{i=1}^{2^n} f(T(i2^{-n})s)$ converges almost everywhere as $n\to\infty$. The key step in the proof is a generalization of the usual ergodic principle that the integral of f over the set $A=\{s|\sup_{0\le n} f_n(s)\ge 0\}$ is positive. Generalized theorems of this sort related to the ergodic theorems of Hopf and Hurewicz, and to the "two-operator" ergodic theorems of the author noted in the preceding review, are also given.

[Author's remark: In theorem 2 the result is established (by the indicated method) only if (S, Ω, μ) is a finite measure space or $\mu(V_n X)=\mu(X)$, $n=0, 1, \cdots$, for $V=T$ and $V=U$.] *J. Schwartz.*

See also: Hewitt and Savage, p. 863; Lorch, p. 879.

Functions of a Complex Variable, Generalizations

★Smirnow, W. I. Lehrgang der höheren Mathematik. Teil III, 2. Deutscher Verlag der Wissenschaften, Berlin, 1955. xi+601 pp. DM 24.80.

This is a translation of Smirnov's Kurs vyssel matematiki, T. III, č. 2 [5th ed., Gostehizdat, Moscow, 1951 (1st ed. 1933)]. Table of contents: I. Elements of complex function theory. II. Conformal mapping and applications. III. Applications of residues. Meromorphic and entire functions. IV. Functions of several complex variables and functions of matrices. V. Linear differential equations. VI. Special functions of mathematical physics. Although much of the material is standard and is to be found in many well-known texts, there are some unusual topics: e.g., Cauchy integrals (following Privaloff), analytic functions of one or more matrices, application of residues to linear differential equations with constant coefficients, application of analytic functions to the wave equation. Throughout the emphasis is on conveying information as rapidly and as easily as possible, and a vast amount of information is so conveyed. The developments are mathematically sound but occasionally appeal to intuition (e.g. with regard to the topology arising in function theory) and some proofs are omitted (e.g., that of the Riemann mapping theorem, because the author considers it of little value for construction of the mapping). On page 30, the Casorati-Weierstrass theorem is attributed to Sochotzki [Sohockif]. *W. Kaplan* (Ann Arbor, Mich.).

Ankeny, N. C.; and Rivlin, T. J. On a theorem of S. Bernstein. Pacific J. Math. **5** (1955), 849–852.

Let $p(z)$ denote a polynomial of degree n, and put $m(R)=\max_{|z|=R}|p(z)|$. A theorem of S. Bernstein states that $m(R)\le R^n m(1)$ if $R>1$. The authors show that this inequality can be refined to (i) $m(R)\le\frac{1}{2}(1+R^n)m(1)$ $(R>1)$ if $p(z)$ is assumed to have no roots within the unit circle. If, on the other hand, all roots of $p(z)$ lie within the unit circle, and if, moreover, $|p(1)|=m(1)$, then (i) does not hold, not even in an interval $1<R<1+\delta$.

N. G. de Bruijn (Amsterdam).

Barhin, G. S. On the degree of stability for quasipolynomials. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 95–97. (Russian)

This note contains a brief discussion of the connection between stability problems for automatic regulation with time-lags and the Routh-Hurwitz problem for exponential polynomials. Some particular results are cited.

R. Bellman (Santa Monica, Calif.).

Labazin, V. G. Some methods for investigating the distribution of roots of transcendental equations. Vestnik Leningrad. Univ. **1953**, no. 2, 3–15. (Russian)

The Routh-Hurwitz problem of determining when the zeros of a function have negative real parts is discussed for some special but important classes of functions, using the method of the variable parameter. The first function is $P(z)\cosh z+tQ(z)\sinh z$, the second is $P(z)\cosh tz+Q(z)\sinh tz$, and the third is $(az+b)-(cz+d)e^{-zt}$.

R. Bellman (Santa Monica, Calif.).

Labazin, V. G. Some properties of roots of transcendental equations which are large in absolute value. Vestnik Leningrad. Univ. **1953**, no. 11, 13–23. (Russian)

The author discusses the asymptotic behavior of the roots of the equation $P(z)e^z+Q(z)e^{-z}+R(z)=0$, where P, Q, and R are polynomials, as $|z|$ approaches infinity.

R. Bellman (Santa Monica, Calif.).

Leont'ev, A. On a class of functions defined by series of Dirichlet polynomials. Moskov. Gos. Univ. Uč. Zap. **145**, Mat. 3 (1949), 3–58. (Russian)

This is the author's thesis, which was summarized in

Uspehi Mat. Nauk (N.S.) **3** (1948), no. 4(26), 176–180 [MR **10**, 364]; some of the material is reproduced in modified form in the author's book, Series of Dirichlet polynomials and their generalizations [Trudy Mat. Inst. Steklov., vol. **39**, 1951; MR **14**, 1074]. Among the things not discussed in the summary are questions of overconvergence; and generalizations of the main results to the case where not all the exponents λ_n are distinct and one considers summands of the form $x^j e^{\lambda_n x}$ as well as $e^{\lambda_n x}$.
R. P. Boas, Jr. (Evanston, Ill.).

Herzog, Fritz; and Piranian, George. Some point sets associated with Taylor series. Michigan Math. J. **3** (1955–56), 69–75.

The authors prove: Let E be a set of type F_σ on C: $|z|=1$. Then there exists a function $f(z)=\sum a_m z^m$ (regular in $|z|<1$) with the following properties: (i) at each point of E, the sequence $\{s_m(z)\}$ converges, where $s_m(z)=\sum_{k=0}^{m} a_k z^k$; (ii) at each point of $C-E$, the sequence $\{s_m(z)\}$ is unbounded. Further, they remark that if E is of measure 2π, then the function $f(z)$ can be constructed in such a way that $\sum |a_m|^2<\infty$. In their construction, the polynomials $[(z^n-1)/(z-1)]^2$ are used skillfully. Some applications and extensions of the theorem are stated.
K. Noshiro (Nagoya).

Bagemihl, F.; and Seidel, W. Functions of bounded characteristic with prescribed ambiguous points. Michigan Math. J. **3** (1955–56), 77–81.

Let $f(z)$ be a complex-valued uniform function defined for $|z|<1$. A point ζ is called an ambiguous point for $f(z)$, if $|\zeta|=1$ and there exist two Jordan arcs J_1 and J_2, terminating in ζ and lying, except for ζ, in $|z|<1$, such that $\lim_{\substack{z\to\zeta \\ z\in J_1}} f(z)$ and $\lim_{\substack{z\to\zeta \\ z\in J_2}} f(z)$ both exist and are unequal. Recently Bagemihl [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 379–382; MR **16**, 1095] has proved that even if no further conditions are imposed on $f(z)$, there are at most enumerably many ambiguous points for $f(z)$. It is well-known that if $f(z)$ is regular and bounded in $|z|<1$, there are no ambiguous points for $f(z)$. In the present paper, the authors prove the following remarkable theorem: Let $E=\{\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots\}$ be an enumerable set of points on $|z|=1$. Then there exists a function $f(z)$, regular and of bounded characteristic in $|z|<1$, for which every element of E is an ambiguous point.
K. Noshiro (Nagoya).

Lohwater, A. J.; and Piranian, G. On a conjecture of Lusin. Michigan Math. J. **3** (1955–56), 63–68.

A point $e^{i\theta}$ is called a Lusin point of the function $f(z)$ provided $f(z)$ is holomorphic in $|z|<1$ and maps every disc $|z-te^{i\theta}|<1-t$ $(0<t<1)$ upon a region (possibly many-sheeted) of infinite area. A conjecture of Lusin [Dokl. Akad. Nauk SSSR (N.S.) **56** (1947), 447–450; MR **9**, 181] may be stated as follows: There exists a bounded function for which every point of $|z|=1$ is a Lusin point. The authors prove: Let $\{a_k\}$ be a sequence of complex numbers different from zero. Then, there exists a set of functions $N_1=N_1(a_1)$, $N_2=N_2(a_1, a_2; n_1)$, $\cdots$, $N_k=N_k(a_1, a_2, \cdots, a_k; n_1, n_2, \cdots, n_{k-1})$, $\cdots$ such that every point $e^{i\theta}$ is a Lusin point of the function $\sum a_k z^{n_k}$, provided $n_k>N_k$ $(k=1, 2, \cdots)$. From this theorem follows that there exists a function $f(z)$, continuous in $|z|\leqq 1$, for which every point $e^{i\theta}$ is a Lusin point. Furthermore, they prove: There exists a function $f(z)=\sum a_n z^n$, holomorphic in $|z|<1$, continuous in $|z|\leqq 1$, and such that (i) $\sum n|a_n|^2=\infty$; (ii) $f(z)$ has no Lusin points on $|z|=1$; (iii) no value is taken infinitely often by $f(z)$ in $|z|\leqq 1$.
K. Noshiro.

Alenicyn, Yu. E. On a method of Nehari. Vestnik Leningrad. Univ. **1955**, no. 2, 71–79. (Russian)

In an earlier paper [Amer. J. Math. **73** (1951), 78–106; MR **12**, 491], the reviewer developed a method for solving a general class of extremal problems within the family of single-valued analytic functions which are regular and bounded in a given multiply-connected domain D. In this method, the solution of the given extremal problem is shown to be dependent on finding, or at least characterizing in certain ways, the extrema of certain auxiliary functions of a finite number of real variables $\lambda_1, \cdots, \lambda_n$. These auxiliary functions are of the form $g(\lambda_1, \cdots, \lambda_n)=\int_C |P(s, \lambda_1, \cdots, \lambda_n)|ds$, where s is the length parameter on the boundary C of D, and P is linear in $\lambda_1, \cdots, \lambda_n$ and continuous in s. The author objects to the fact that the conditions following from the vanishing of the first variation of g are written down without carrying out the intermediate elementary computations. Saying that these steps have to be justified, he then proceeds to justify them in a few lines. More serious is his objection to the one case in which the second variation of g is used (in order to characterize the extremum in question as a minimum). He shows by appropriately constructed examples that in the second variation of functions of this type there may appear terms which depend on the location of the zeros of P on C, and which may be difficult to evaluate. Contrary to his belief, however, this objection does not invalidate the function-theoretic result in question. In this case the existence of a non-zero minimum (which he questions) follows from elementary function-theoretic considerations, and the use of the second variation can be altogether dispensed with.

On the positive side, the author shows that the extremal functions obtained by the reviewer's method are all unique up to a trivial transformation.
Z. Nehari (Pittsburgh, Pa.).

Babuška, Ivo. Über eine gewisse Eigenschaft der harmonischen Funktionen. Czechoslovak Math. J. **5(80)** (1955), 220–233. (Russian. German summary)

The author considers domains Ω whose boundary C is a simple orientable curve which consists of finitely many, say N, arcs (A_kA_{k+1}); C possesses the following property: Let $\vartheta(s)$ denote the angle between the positive direction of the tangent to C and the x-axis. Then on every arc (A_kA_{k+1}), $[d\vartheta(s)/ds]$ is a continuous function of s for which $\int_{(A_kA_{k+1})} |d\vartheta(s)/ds|^p ds<\infty$ $(p>1)$. At the endpoints A_k of the arcs we have $|\vartheta(A_k)_+ - \vartheta(A_k)_-|\neq 0 \pmod{\pi}$ $(k=1, 2, \cdots, N; N<\infty)$. Generalizing the results of his previous paper [Časopis Pěst. Mat. **79** (1954), 41–63; MR **16**, 1109], the author shows that for every function φ holomorphic in Ω, which satisfies the relation $\operatorname{Im}\varphi(Z_0)=0$, $Z_0\in\Omega$, the inequality

$$\iint_\Omega |\varphi|^{2-\varepsilon}dxdy \leqq K(\varepsilon)\left[\iint_\Omega (\operatorname{Re}\varphi)^2 dxdy\right]^{1-\frac{\varepsilon}{2}}$$

holds; here $0<\varepsilon<1$, and $K(\varepsilon)<1$ depends upon the domain and the quantity ε (but is independent of φ).

The above results have applications in the theory of orthogonal functions and in the theory of conjugate harmonic functions.
S. Bergman (Stanford, Calif.).

Waterman, Daniel. On functions analytic in a half-plane. Trans. Amer. Math. Soc. **81** (1956), 167–194.

The author studies the class $\mathcal{H}_p$ of functions $\phi(s)$

regular in the half plane $\sigma>0$, where $s=\sigma+i\tau$, and for which

$$\|\phi\|_p=\sup_{0<\sigma<\infty}\int_{-\infty}^{+\infty}|\phi(\sigma+i\tau)|^p d\tau<+\infty.$$

Let Ω_t be the angle given by $|\tau-t|<k\sigma$, where k is a fixed positive number and let

$$s(\tau)=\left\{\iint_{\Omega_\tau}|\phi'(s)|^2 d\omega\right\}^{\frac{1}{2}}$$

(author's notation!). Then if $p\geqq 1$

$$\int_{-\infty}^{+\infty}|s(\tau)|^p d\tau\leqq A(p,k)\|\phi\|_p,$$

and if $p>1$

$$\|\phi\|_p<A(p,k)\int_{-\infty}^{+\infty}|s(\tau)|^p d\tau.$$

The corresponding results for two functions g and g^* instead of s are also proved.

These results are analogues of theorems proved for functions in the unit circle by Littlewood and Paley [Proc. London Math. Soc. (2) **42** (1936), 52–89] and others. There are a number of rather vague references to previous work including two to an unpublished manuscript of E. Trombley. One of the results attributed to him (Lemma 6) is in fact a special case of a theorem of Hardy and Littlewood [Acta Math. **54** (1930), 81–116, p. 98, Th. 10]. *W. K. Hayman* (Exeter).

Hersch, Joseph. Longueurs extrémales et théorie des fonctions. Comment. Math. Helv. **29** (1955), 301–337.

This thesis is a contribution to the theory of extremal length of Ahlfors and Beurling. The principal results have been announced previously [Hersch, C. R. Acad. Sci. Paris **235** (1952), 569–571; **237** (1953), 641–643; Hersch and Pfluger, ibid. **237** (1953), 1205–1207; MR **14**, 262; **15**, 115, 301]. The author introduces a modified definition of extremal length in which the double integrals of the usual definition are replaced by upper Darboux integrals and the line integrals are replaced by lower integrals. The condition that the competing curves be rectifiable is dropped. The modified definition is introduced with the view of simplifying the usual demonstrations of the theory. In addition, it plays a role in the author's "Contribution à la théorie des fonctions pseudo-analytiques" [Comment. Math. Helv. **30** (1956), 1–18; MR **17**, 357]. The standard properties of extremal length are developed in the first chapter. In the appendix to the first chapter the notion of a numerical family (non-negative real homogeneous function defined on a set of "broken curves") is introduced and a module which appears as a generalization of the reciprocal of the extremal length is associated with a numerical family. A number of properties of the module of a numerical family are established. Chapter II is concerned with the relation of harmonic measure and hyperbolic distance with extremal length. Chapter III is concerned with applications. These include sharpened forms of known comparison theorems for harmonic measure (Carleman) and hyperbolic distance (for the latter the regions G_1, G_2 compared are simply-connected and satisfy $\bar{G}_1\subset G_2$). A generalization of the Phragmén-Lindelöf principle is given. A number of estimates are established (for hyperbolic distance § 5; harmonic measure and modules § 6). The paper is concluded with outlines of proofs of theorems of Koebe and Ahlfors. *M. Heins* (Providence, R.I.).

Falgas, Maurice. Sur certaines fonctions associées aux bases de polynomes et leur utilisation à la définition des séries de base et à l'étude de l'effectivité de ces bases. I. La définition des séries de base. C. R. Acad. Sci. Paris **242** (1956), 1563–1566.

Falgas, Maurice. Sur certaines fonctions associées aux bases de polynomes de leur utilisation à la définition des séries de base et à l'étude de l'effectivité de ces bases. II. L'effectivité des bases de polynomes. C. R. Acad. Sci. Paris **242** (1956), 1677–1679.

Let $\{p_n(z)\}$ be a basic set of polynomials [in the sense of Whittaker, Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; MR **11**, 344]; let $(z-z_0)^n=\sum_{m=0}^{\infty}\pi_{nm}(z_0)p_m(z)$. Suppose that

$$\sum_{n=0}^{\infty}\pi_{nm}(z_0)(x-z_0)^{n-1}$$

converges in a neighborhood of ∞; it then converges to an analytic function $\phi_m(x)$ which is independent of z_0. The author uses the $\phi_m(x)$ and Faber polynomials to generalize various notions and theorems of the Whittaker theory to domains much more general than the conventional circular disks. The coefficients of the basic series are defined in a general way. The author introduces variations on Whittaker's definition of effectiveness and gives necessary and sufficient conditions for the existence of the basic series and for it to possess various kinds of effectiveness. *R. P. Boas, Jr.* (Evanston, Ill.).

Erwe, Friedhelm. Eine Interpolationsaufgabe. Arch. Math. **7** (1956), 55–58.

Soit $f(z)$ une fonction holomorphe au voisinage de l'origine, $\{z_n\}$ une suite tendant vers zéro, et telle que, M désignant un nombre positif, on ait: $|z_{n+1}|\leqq M|z_n|^2$, pour n assez grand; alors si $f^{(n)}(z_n)=0$, pour presque toutes les valeurs de n, $f(z)$ est un polynome.

J. Favard (Grenoble).

Pirl, Udo. Isotherme Kurvenscharen und zugehörige Extremalprobleme der konformen Abbildung. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **4** (1955), 1225–1251.

The author considers the following extremal problem: Let B_n be a plane domain of connectivity n. Let F_ν $(\nu=1, 2, \cdots, n)$ be non-overlapping doubly-connected domains lying in B_n and respectively homotopic to the boundary components (supposed non-degenerate) of B_n. Let F_ν have conformal module M_ν (the ratio >1 of the radii of a circular ring conformally equivalent to F_ν). Let $\alpha_\nu>0$. The problem is then to determine such a configuration where $\prod_{\nu=1}^{n}M_\nu^{\alpha_\nu}$ is maximal. The author treats the problem by using boundary identification of doubly-connected domains to obtain domains B_n in which the extremal configuration is almost evident and then applies a continuity argument to deduce that all domains B_n are obtained by this process. The author treats the case $n=3$ in great detail and indicates the procedure for general n. The results obtained are special cases of a general principle discovered some years ago by the reviewer and mentioned in numerous papers [Amer. J. Math. **75** (1953), 510–522; Duke Math. J. **21** (1954), 155–162; Trans. Amer. Math. Soc. **76** (1954), 389–396; **78** (1955), 510–515; Ann. of Math. (2) **59** (1954), 490–504; MR **15**, 115, 694; **16**, 24, 684; **15**, 786]. In particular, the above results for $n=3$ are subsumed under Lemma 3 (p. 513) of the first-mentioned paper. As the reviewer understands the author's prescription for construction for general

n, simple examples show that it will not provide all plane domains of connectivity n. If this is so, the proof in this case is defective. The author gives certain characterizations of the extremal configurations but does not recognize that each is associated with a quadratic differential (in his special case, positive). When this is known the author's remark in § 11 becomes a very simple special case of Teichmüller's general result on extremal quasiconformal mappings. The author also indicates some extensions when boundary components of B_n are allowed to reduce to a point. *J. A. Jenkins.*

Gerber, Robert. Sur un lemme de représentation conforme. C. R. Acad. Sci. Paris **241** (1955), 1440–1441.

Let D be a domain in the z-plane bounded by a Jordan curve Γ and of finite area σ. Let α_1, α_2, α_3, α_4 be four consecutive points on Γ and let Λ be the length of the shortest curve in D which connects the arcs $\widehat{\alpha_2\alpha_3}$ and $\widehat{\alpha_1\alpha_4}$ of Γ. If we map D onto the upper half of the plane $t+i\tau$ we denote by t_i the image of α_i and by r the cross-ratio $[t_3, t_2, t_4, t_1]$. R. Huron [Ann. Fac. Sci. Univ. Toulouse (4) **15** (1951), 155–160; MR **14**, 549] proved the inequality: $\Lambda^2\sigma^{-1}\leqq\prod \log r^{-1}$ for $0\leqq r\leqq 1$. J. Jenkins [J. London Math. Soc. **30** (1955), 382–384; MR **17**, 251]; showed that this result is an easy consequence of the general method of extremal metrics due to Ahlfors and Beurling. The author uses Jenkins' auxiliary mapping but derives the result directly from the Schwarz inequality. Since the general lemma used by Jenkins is proved in exactly the same way the present proof seems identical with that of Jenkins. *M. Schiffer* (Stanford, Calif.).

Walsh, J. L.; and Rosenfeld, L. On the boundary behavior of a conformal map. Trans. Amer. Math. Soc. **81** (1956), 49–73.

Etude du comportement, à l'infini, de la fonction $w=f(z)$ qui représente conformément la bande $|y|<\pi/2$ du plan $z=x+iy$, sur une bande S du plan $w=u+iv$. Les A. utilisent essentiellement la théorie de Carathéodory (représentation conforme des domaines variables). Ce même problème avait déjà été étudié par S. E. Warschawski [mêmes Trans. **51** (1942), 280–335; MR **4**, 9] et J. Lelong-Ferrand [J. Analyse Math. **2** (1952), 51–71; MR **14**, 742], ce dernier mémoire utilisant déjà la théorie de Carathéodory. Mais les hypothèses faites ici sont différentes et sensiblement intermédiaires entre celles de Warschawski et Lelong-Ferrand, tant en ce qui concerne la régularité de la frontière que son allure globale; et la méthode de Carathéodory est utilisée différemment. Un large usage est fait de la propriété (B):

$$\varphi[U\varphi(u)+u]/\varphi(u)\to 1$$

uniformément pour $|U|<U_0$ quel que soit U_0.

Citons entre autres résultats: Si S est limitée par $v=\varphi^+(u)$ et $v=\varphi^-(u)$ (φ^+ et φ^- pouvant être multiformes), $-\infty<u_1<u<+\infty$, $\varphi^+(u)>0$ satisfaisant à (B) et à $\varphi^-(u)/\varphi^+(u)\to\lambda$ $(0\leqq\lambda<1)$; si de plus $u(x+iy)\to+\infty$ quand $x\to+\infty$, si on pose $\theta(u)=\varphi^+(u)-\varphi^-(u)$ et si $x_n\to+\infty$, alors $[f(z+x_n)-f(x_n)]/\theta[u(x_n)]\to z/\pi$ uniformément sur tout compact contenu dans S. De plus si on pose $z_n=x_n+iy_n$ et si $\limsup|y_n|<\pi/2$, alors

$$\frac{f'(z_n)}{\theta[u(z_n)]}\to\frac{1}{\pi}\cdot\frac{f^{\nu}(z_n)}{\theta[u(z_n)]}\to 0 \quad (\nu\geqq 2),\ \arg f'(z_n)\to 0.$$

Les résultats obtenus sont adaptés à l'étude de la représentation conforme au voisinage d'un point à distance finie. *J. Lelong* (Lille).

Terzioğlu, A. Nazim; and Kahramaner, Suzan. Ein Verzerrungssatz des Argumentes der schlichten Funktionen. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 81–90. (Turkish summary)

Let $f(z)=z+a_2z^2+\cdots$ be regular and univalent in $|z|<1$. The authors prove that for $|z|=r$, $|\arg f(z)/z|\leqq Cr/(1-r)^2$ with $|a_2|\leqq C\leqq\log 16$. However, the precise bound $\log(1+r)/(1-r)$ was obtained earlier by Goluzin [Mat. Sb. N.S. **2(44)** (1937), 37–64, 685–688]. *A. W. Goodman* (Lexington, Ky.).

Diamantopoulos, Th. Geometrische Interpretation der höheren Ableitungen und der Reziprokanten bei der konformen Abbildung der Ebene. Arch. Math. **7** (1956), 67–73.

Les „Reziprokanten" d'une représentation conforme $w(z)$ sont les fonctions rationnelles des dérivées successives $w'(z)$, $w''(z)$, $\cdots$ qui ne subissent qu'un changement de signe quand les dérivées sont remplacées par leurs inverses. Les „Schrumpfungsradien" d'une courbe $z(s)$ et de son image par $w(z)$, déjà définis par l'A. [Bull. Soc. Math. Grèce **28** (1954), 81–100; MR **16**, 166] semblent liés à une représentation paramétrique commune. Utilisant le paramètre $\alpha=\int\sqrt{w'(z)}\,dz$, l'A. interprète les "Reziprokanten" de la forme $d^n\log w'(z)/d\alpha^n$ comme des polynomes rationnels homogènes de vecteurs définis au moyen des rayons de courbure et des "Schrumpfungsradien" des divers ordres des deux courbes. Inversement la connaissance de tous ces rayons et de dérivée $w'(z)$ en un point détermine la représentation $w(z)$ dans un voisinage de ce point. *J. Lelong* (Lille).

Yûjôbô, Zuiman. On absolutely continuous functions of two or more variables in the Tonelli sense and quasiconformal mappings in the A. Mori sense. Comment. Math. Univ. St. Paul. **4** (1955), 67–92.

A topological mapping $w=T(z)$, $w=u+iv$, $z=x+iy$ is called K-quasiconformal (K-QC), if the quotient of the conformal moduli m and m' of any quadrilateral (=topological square with two distinguished opposite sides) is $m'/m\leqq K$ [Pfluger, Ann. Inst. Fourier, Grenoble **2** (1950), 69–80; MR **13**, 453; Ahlfors, J. Analyse Math. **3** (1954), 1–58, 207–208; MR **16**, 348]. After having proved that absolute continuity in the Tonelli sense of a function of two variables is invariant under a topological mapping with components $u(x, y)$, $v(x, y)$, $x(u, v)$ and $y(u, v)$ which satisfy the Lipschitz condition, the author establishes in four theorems necessary and sufficient conditions for a topological mapping to be quasiconformal (necessity proved by A. Mori [Sûgaku **7** (1955), 75–89 (Japanese)]). Example: T is K-QC if and only if (i) u and v are totally differentiable almost everywhere,

$$\text{(ii)}\ \max_{\theta}|u_x\cos\theta+u_y\sin\theta+i(v_x\cos\theta+v_y\sin\theta)|^2\leqq K\frac{\partial(u, v)}{\partial(x, y)}$$

almost everywhere and (iii) there exist two straight lines not parallel to each other, say a and b, such that on almost all the straight lines of the plane which are either parallel to a or to b the functions $u(x, y)$ and $v(x, y)$ satisfy Lusin's condition (N). Some auxiliary theorems, where the notion of modulus does not enter, are stated for n-dimensional Euclidean space. There is a connection with some theorems of the reviewer [Proc. Amer. Math. Soc. **6** (1955), 903–909; MR **17**, 473]. *K. Strebel* (Fribourg).

Künzi, Hans. Zur Theorie der Viertelsenden Riemannscher Flächen. Comment. Math. Helv. **30** (1956), 107–115.

Construction of Riemann surfaces whose "Streckenkomplex" contains two or four quadrants of rectangular nets. It is characteristic for the investigation that the branch-points differ from quadrant to quadrant. The order of the corresponding function and the ramification indices are derived. The method is the same as in earlier work by the author [same Comment. **29** (1955), 223–257; MR **16**, 1095]. *L. Ahlfors* (Cambridge, Mass.).

Osserman, Robert. A hyperbolic surface in 3-space. Proc. Amer. Math. Soc. **7** (1956), 54–58.

The following problem, originally raised by C. Loewner, was verbally communicated by L. Bers to the Conference on Riemann Surfaces in Princeton, 1951: Does there exist a surface of the form $z=f(x, y)$ which covers every point of the (x, y)-plane exactly once, and which is conformally equivalent to the unit disk? That the answer is affirmative, was announced by the author in the Proc. Internat. Congress Math., Amsterdam, 1954, vol. II, Noordhoff, Groningen, 1954, p. 153. In the present paper he gives the construction of a surface with the required properties. *L. Sario* (Los Angeles, Calif.).

★ **Bers, Lipman. Local theory of pseudoanalytic functions.** Lectures on functions of a complex variable, pp. 213–244. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Pseudo-analytic functions originally were conceived as the solutions of elliptic systems of linear, first-order partial differential equations in two unknowns and in two independent variables. In canonical form, these equations are

$$(*) \qquad w_{\bar{z}}=aw+b\bar{w},$$

where $w=u+iv$ represents a complex-valued function, a and b being complex-valued coefficients, and where $\partial/\partial\bar{z}=\frac{1}{2}[(\partial/\partial x)+i(\partial/\partial y)]$. In the original theory [cf., e.g., Bers, Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ., 1953; MR **15**, 211], a and b are assumed "smooth" (e.g., Hölder-continuously differentiable); it follows that there exist two independent, smooth, particular solutions F and G of (*) such that at any point $z_0=x_0+iy_0$ the validity of equation (*) for a continuously differentiable function w is equivalent to the existence of

$$\dot{w}\equiv\frac{d_{(F,G)}w}{dz}\equiv\lim_{z\to z_0}(z-z_0)^{-1}[w(z)-\phi(z_0)F(z)-\psi(z_0)G(z)]$$

$\phi(z)$ and $\psi(z)$ being defined by the requirement that $w(z)=\phi(z)F(z)+\psi(z)G(z)$. The solutions F and G are called a generating pair; any function w for which $\dot{w}$ exists is called an (F, G)-pseudo-analytic function (of the first kind), $\dot{w}$ its (F, G)-derivative, and ${}_*w\equiv\phi+i\psi$ an (F, G)-pseudo-analytic function of the second kind. ${}_*w$ satisfies a second elliptic differential equation

$$(**) \qquad w_{\bar{z}}=\frac{F+iG\bar{w}}{F+iGz}.$$

The present paper is devoted to a generalized local theory applicable when the generating pair are both Hölder-continuous. Then equation (*) need not be defined, and w need not be differentiable, but ${}_*w$ must be differentiable and must, as before, satisfy (**), and many close parallels to the theory of analytic functions still obtain as in the earlier theory. Some of the principal ones are listed below. First, ${}_*w$ is an interior transformation, and moreover, its critical points are precisely the points in the neighborhood of which the mapping $\zeta=w(z)$ is not one-to-one. Secondly, the isolated singularities of a single-valued pseudo-analytic function, if not removable, are either poles or essential singularities. Specifically, if w is bounded in the neighborhood of an isolated singularity z_0, then w is regular at z_0; if w is not bounded near z_0 but, for some $N>0$, (†) $\lim_{z\to z_0}(z-z_0)^N w(z)=0$, then for a suitable integer n and a complex number $a\neq 0$ $w(z)\sim a(z-z_0)^{-n}$ and $\dot{w}\sim -na(z-z_0)^{-n-1}$ as $z\to z_0$; if w is not bounded and (†) holds for no N, then Weierstrass' theorem applies to the behavior of w in the neighborhood of z_0 and Picard's theorem to the behavior of ${}_*w$. A third parallel to function theory is the following remarkable fact: every (F, G)-derivative of an (F, G)-pseudo-analytic function is again a pseudo-analytic function but with respect to another pair of generators, say F_1, G_1, which again will be Hölder-continuous. Moreover, every (F_1, G_1)-pseudo-analytic function is the (F, G)-derivative of an (F, G)-pseudo-analytic function. The pair (F_1, G_1) is called a "successor" of (F, G), (F, G) its "predecessor." Every generating pair has a predecessor as well as a successor. With the help of a predecessor pair, a theorem of Morera's type is stated. If (F_{n+1}, G_{n+1}) is a successor to (F_n, G_n), $n=0, 1, \cdots$ $(F_0=F, G_0=G)$, and if

$$w^{[n+1]}=\frac{d_{(F_n,G_n)}w^{[n]}}{dz},$$

then the sequence $w^{[n]}(z_0)$ determines the pseudo-analytic function $w(z)$ uniquely. *A. Douglis* (New York, N.Y.).

Sce, Michele. Sui sistemi di equazioni a derivate parziali inerenti alle algebre reali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 32–38.

In a previous paper [same Rend. (8) **16** (1954), 30–35; MR **16**, 786] the author defined monogenic function and showed that if $y=\sum y_i u_i$ is a function of $x=\sum x_i u_i$ in a linear associative algebra A (u_i=basis elements of A), then the y_i satisfy a set of partial differential equations of the first order with constant coefficients. In the present paper he shows that the structure of the algebra A (and not the basis) determines whether the system of differential equations is elliptic, parabolic, or hyperbolic.

J. A. Ward (Holloman, N.M.).

Moisil, Gr. C. Sur une formule de moyennes. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 52–54. (Romanian. Russian and French summaries)

The author shows that certain equations in the theory of elasticity and in the theory of incompressible fluids may be obtained from certain functions analytic in the sense of Federov [Mat. Sb. N.S. **18(60)** (1946), 353–378; MR **8**, 25] in the hypercomplex number system of Sobrero [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **6** (1934), 1–64].

J. A. Ward (Holloman, N.M.).

Analytic Functions of Several Variables

Džrbašyan, M. M. On the theory of some classes of entire functions of several variables. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauk **8** (1955), no. 4, 1–23. (Russian. Armenian summary)

The author defines classes of entire functions of two

complex variables by means of order and type in each variable separately, and gives generalizations of the familiar relations among order, type, maximum modulus, and coefficients; also of the Borel transform and the Pólya integral representation. *R. P. Boas, Jr.*

Temlyakov, A. A. Entire functions of two complex variables. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 7–16. (Russian)

Let x, y, u, v be complex variables satisfying the relations $u=x+ye^{it}$, $v=ue^{-it}=y+xe^{-it}$, where t is a real variable. The analytic functions

$$F(x, y)=\sum a_{mn}x^m y^n,$$

$$f_1(u, v)=\sum \frac{m!n!}{(m+n)!}a_{mn}u^m v^n=f(u, e^{-it})$$

are connected by the relations

$$f(u, e^{-it})=\int_0^1\Big(\frac{d}{du}uF(uT, u(1-T)e^{-it})\Big)dT,$$

$$F(x, y)=\frac{1}{2\pi}\int_0^{2\pi} f(u, e^{-it})dt=$$

$$\frac{1}{2\pi}\int_0^{2\pi} f_1(x+ye^{it}, y+xe^{-it})dt.$$

The series $F(x, y)$ is convergent in the hypercone $|x|+|y|<R$, if and only if $f_1(u, v)$ converges in the bicylinder $|u|<R$, $|v|<R$. The author considers the case where $F(x, y)$ is integral. The maximum of $|F(x, y)|$ on $|x|+|y|=R$ is denoted $M(R)$ and the order $\lambda(F)=\limsup \log\log M(R)/\log R$ and the type $\sigma(F)=\limsup \log M(R)/R^{\lambda(F)}$ are introduced. The author proves that these numbers coincide with the order and type of $f(u, e^{-it})$ for fixed t. *H. Tornehave* (Copenhagen).

Temlyakov, A. A. Integral representation of analytic functions of two complex variables. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **21** (1954), 7–22. (Russian)

Let p and q be relative primes. Let $f(\tau)$ be a real-valued function, positive and differentiable when $0<\tau<1$ and satisfying the conditions

$$f'(\tau)<0, \frac{d}{d\tau}(\tau^p/f(\tau))>0, \lim_{\tau=0}\tau^p/f(\tau)=0,$$

$$\lim_{\tau=1}\frac{d}{d\tau}\log(\tau^p/f(\tau))>0.$$

Let D be a domain in the complex (w, z)-space, let the boundary of D be the hypersurface

$$|w|=r_1(\tau)=\tau^p/f(\tau);$$

$$|z|=r_2(\tau)=(1-\tau)^q\exp\left(-p^{-1}q\int_0^\tau \tau(1-\tau)^{-1}d\ln f(\tau)\right);$$

$$0\leqq\tau\leqq 1,$$

and let $F(w, z)$ be a function defined by a power series $\sum_{k,l=0}^{\infty} a_{kl}w^{k/p}z^{l/q}$ convergent in the closure of D. Let C denote the curve $|\zeta|=1$, $\eta=e^{-it}\zeta$ and

$$u=\tau(w/r_1(\tau))^{1/q}+(1-\tau)(z/r_2(\tau))^{1/q}e^{it}.$$

Under these conditions the author proves the integral representation

$$F(w; z)=\frac{1}{4\pi^2 i}\int_0^{2\pi}dt\int_0^1 d\tau\int_C\frac{\zeta F(r_1(\tau)\zeta^p, r_2(\tau)\eta^q)}{(\zeta-u)^2}d\zeta.$$

The integral is obviously extended over the boundary of D, which is a non-analytic surface. The formula is a generalization of Cauchy's formula and the author derives corresponding generalizations of Poisson's formula and Cauchy's inequalities. *H. Tornehave* (Copenhagen).

Harmonic Functions, Potential Theory

Pernavs, Nora. On a mixed boundary value problem of harmonic functions. Proc. Amer. Math. Soc. **7** (1956), 127–130.

Let D be a domain in the xy-plane which is bounded by a finite number of closed curves C, and denote by s a finite set of disjoint subarcs of C. The mixed boundary-value problem treated by the author consists in showing that there exists a harmonic function in D which has given values on s and whose normal derivatives assume given values on $C-s$. The existence proof is carried out by means of a modified form of Perron's method for the solution of the Dirichlet problem. In one step of the proof, however, the author cannot avoid the use of the conformal mapping of a general simply-connected domain onto a certain canonical domain, and this excludes the extension of the method to harmonic functions of more than two variables. *Z. Nehari* (Pittsburgh, Pa.).

Di Palo, Raffaele. Sopra una forma assoluta del teorema di Green generalizzato. Giorn. Mat. Battaglini (5) **3(83)** (1955), 61–67.

This paper discusses a generalization of Green's Theorem involving in iterated form an operator corresponding to the Laplacian. Author makes extensive use of material for which he cites Burali-Forti and Marcolongo, Analisi vettoriale generale e applicazioni, v. 1 [2nd ed., Zanichelli, Bologna, 1929]. *F. W. Perkins* (Hanover, N.H.).

Nicolescu, Miron. Propriétés de décomposition des fonctions de plus variables réelles et en particulier des fonctions polyharmoniques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **3** (1954), no. 4–5, 53–63. (Romanian. Russian and French summaries)

Let D be a bounded Dirichlet domain in euclidean n-space, let $G(P, Q)$ be the Green's function for D, and let the function $\varphi(P)$ be integrable in D. Let σ_0 be the measure of the unit sphere in n-space, and define $G^1(P, Q)=G(P, Q)$,

$$G^p(P, Q)=\frac{-1}{(n-2)\sigma_0}\int_D G^{p-1}(P, Q_1)G(Q_1, Q)dQ_1,$$

$$G^p\cdot\varphi=\frac{-1}{(n-2)\sigma_0}\int_D G^p(P, Q)\varphi(Q)dQ,$$

$p=2, 3, \cdots$; for $n=2$, the factor $(n-2)$ is replaced by 1. It is shown that every function u, polyharmonic of order p in D, can be expressed in the form

$$u=u_0+G^1\cdot u_1+\cdots+G^{p-1}\cdot u_{p-1},$$

where $u_0, u_1, \cdots, u_{p-1}$ are the harmonic functions uniquely determined in D by the boundary values of $u, \Delta u, \cdots, \Delta^{p-1}u$. Indeed, every function u of class $C^{(2p)}$ in D can be expressed as

$$u=u_0+G^1\cdot u_1+\cdots+G^{p-1}\cdot u_{p-1}+G^p\cdot\Delta^p u,$$

and if u is of class $C^{(\infty)}$ and all its Laplacians have a common bound, then in a sufficiently small sphere u can be expressed in terms of the corresponding infinite series. *E. F. Beckenbach* (Los Angeles, Calif.).

See also: Babuška, p. 834.

Series, Summability

★**Salehov, G. S. Vyčislenie ryadov. [The computation of series.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 143 pp. 3.60 rubles.

This little book is an exposition of useful methods for obtaining numerical approximations to values of convergent series whose terms are given real numbers. In addition to classic methods, it contains many developments given in recent years by Russians. Nearly all of the book is devoted to presentation of methods, there being very few presentations of results of numerical calculations. Perhaps the purpose and nature of the book are best revealed by a section on series of the form $\Sigma[P(n)/Q(n)]x^n$, where P and Q are polynomials. A procedure is given which yields the application

$$\sum_{n=1}^{\infty}\frac{nx^n}{n^2+1}=-\frac{\ln(1-x)}{x}\left[1+\left(1-\frac{1}{x}\right)+\frac{1}{2}\left(1-\frac{1}{x}\right)^2\right]$$
$$-\frac{5}{3}+\frac{7}{4x}-\frac{1}{2x^2}-10\sum_{n=1}^{\infty}\frac{x^n}{(n+1)(n+2)(n+3)(n^2+1)}$$

which is valid when $-1\leq x<1$ and which replaces a series which converges very slowly when x is near -1 or 1 by a series which converges quite rapidly. One could expect that a book of this nature would give the Euler-Maclaurin summation formula and illustrate its exceptional usefulness. This is omitted, perhaps because an adequate treatment would be lengthy, and the chapter on this subject in the German and English editions of Konrad Knopp's book on infinite series should be regarded as an important appendix to the book of Salehov.

R. P. Agnew (Ithaca, N.Y.).

Orloff, Constantin. Une interprétation géométrique des séries alternées. Bull. Soc. Math. Phys. Serbie **7** (1955), 3–10; errata, 136. (Serbo-Croatian summary)

Let $a_0-a_1+a_2-a_3+\cdots$ be a given alternating series for which $a_0>a_1>a_2>\cdots$ and $a_k\to 0$. For each $n=0, 1, 2, \cdots$ let P_n be the point in a plane with rectangular coordinates (x_n, y_n) defined by $x_n=\sum_{2k\leq n}a_{2k}$ and $y_n=\sum_{2k+1\leq n}a_{2k+1}$ so that $(x_0, y_0)=(a_0, 0)$, $(x_1, y_1)=(a_0, a_1)$, $(x_2, y_2)=(a_0+a_2, a_1)$, $\cdots$. The curve consisting of line segments from P_0 to P_2 to P_4 to P_6 to $\cdots$ is a lower characteristic curve, and an upper characteristic curve is similarly determined by $P_1, P_3, P_5, \cdots$. In case $\sum(-1)^k a_k=S$ and $\sum a_k=\infty$, the line $y=x-S$ is an asymptote of these curves. The geometry involved is intended to be useful in the theory of alternating series. Unfortunately, it leads the author to the conclusion that if $a_0-a_1+a_2+\cdots$ converges but does not converge absolutely, then $a_{n+1}/a_n\to 1$ as $n\to\infty$. The conclusion is false.

R. P. Agnew (Ithaca, N.Y.).

Karadžić, Lazar. Nouveaux critères de convergence des séries à termes positifs et la géométrie de Lobatchewsky s'y rattachant. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 34–46.

The author employes the geometry of Lobatschewsky to give new proofs of a number of well-known (rules of Cauchy, d'Alembert and Raabe) convergence criteria for series of positive terms. Other criteria are proven which are closely related to the logarithmic-scales tests. A typical result is the following. If

$$\liminf n\left(1-\frac{a_{n+1}}{a_n}\right)>1 \text{ or if } \liminf\frac{n}{\ln n}(1-a_n^{1/n})>1,$$

the series $\sum a_n$ converges; if

$$\limsup\frac{n}{\ln n}(1-a_n^{1/n})\leq 1 \text{ or if } \limsup n\left(1-\frac{a_{n+1}}{a_n}\right)\leq 1,$$

the series $\sum a_n$ diverges ($a_n>0$).

V. F. Cowling.

Drahlin, E. H. A criterion for convergence of series. Molotov. Gos. Univ. Uč. Zap. **8**, no. 1 (1953), 11–12. (Russian)

It is shown that $\sum_{k=1}^{\infty}F(k)$ is convergent if $F(x)$ is a positive continuous decreasing function which is representable in the form $F(x)=f(x)g(x)$ where $f(x)$ and $g(x)$ are positive, $f(x)$ has a continuous derivative, $f(x)\to 0$ as $x\to\infty$, and there is a constant c for which $f(x)<c$ and $cf(x)g(x)<-f'(x)$. The proof depends upon using the hypotheses to obtain.

$$cf(x)g(x)-[f(x)]^2g(x)<-f'(x),$$

dividing by $c-f(x)$, integrating, and applying the integral test for convergence of series. A similar theorem gives conditions sufficient for divergence of $\sum F(k)$.

R. P. Agnew (Ithaca, N.Y.).

Každan, Z. N. On convergence of series of functions of the form $\varphi(nx)$. Moskov. Gos. Univ. Uč. Zap. **163**, Mat. **6** (1952), 99–122. (Russian)

Let C' be the set of continuous functions $g(x)$ of period one and such that $\int_0^1 g(x)dx=0$. The author investigates which conclusions can be drawn about the series $\sum a_n$ from hypotheses about the behavior of $\sum a_n g(nx)$ for all $g\in C'$. Typical results are: 1. If α is an irrational number and if the partial sums of $\sum_{n=1}^{\infty}a_n g(n\alpha)$ are bounded for every $g\in C'$, then $\sum_{n=1}^{\infty}|a_n|<\infty$. 2. There is a sequence $\{a_n\}$ such that $\sum a_n g(nx)$ converges almost everywhere on $(0, 1)$ for every $g\in C'$, while $\sum|a_n|=\infty$. 3. If for every continuous, periodic function $h(x)$ of period one $\sum a_n h(nx)$ and all its rearrangements converge on a set of the second category ($0<x<1$), then $\sum|a_n|<\infty$. 4. If $k(x)$ is an integrable function on $(0, 1)$ and $k(x+1)=k(x)$ and if $k(x)\neq 0$ almost everywhere, then the absolute convergence of $\sum a_n k(nx)$ entails $\sum|a_n|<\infty$. 5. If $\psi_n\in L^2(0, 1)$ and the series $\sum c_n\psi_n(x)$ and all its rearrangements converge in the mean, then $\sum|c_n|^2\|\psi_n\|^2<\infty$. This result was published, with a simpler proof, by Orlicz [Studia Math. **4** (1933), 33–37].

W. H. J. Fuchs.

Alhimov, M. On summation of a class of double sequences. Dnepropetrov. Gos. Univ. Nauč. Zap. **41** (1953), 137–143. (Russian)

A double sequence $S_{j,k}$ is said to be horizontally (vertically) convergent to S_h (S_v) is there exists a positive function $\phi(x)$ such that $\phi(x)\to\infty$ as $x\to\infty$ and to each $\varepsilon>0$ there corresponds an integer N such that $|S_{jk}-S_h|<\varepsilon$ ($|S_{jk}-S_v|<\varepsilon$) whenever $k>N$ and $j>\phi(k)$ ($j>N$ and $k>\phi(j)$). If $S_h=S_v=S$, the sequence is said to be laterally convergent to S. The sequence S_{jk} is said to be horizontally (vertically) bounded if there exist constants M and N and a positive function $\phi(x)$ such that $\phi(x)\to\infty$ as $x\to\infty$ and $|S_{j,k}|<M$ whenever $k>N$ and $j>\phi(k)$ ($j>N$ and $k>\phi(j)$). Let $A=(a_{nk})$ and $B=(b_{nk})$ be regular triangular matrices with elements defined when $0\leq k\leq n<\infty$ and let

$$\sigma_{m,n}=\sum_{j,k=0}^{m,n} a_{mj}b_{nk}S_{jk}$$

so that S_{jk} is evaluable to S by the factored transformation AB if $\sigma_{mn}\to S$ as $m,n\to\infty$. It is shown that if S_{jk} is horizontally convergent to S and the AB transform σ_{mn} is horizontally bounded, then σ_{mn} is horizontally convergent to S. Analogous theorems apply to vertical and lateral convergence. These theorems are analogous to theorems of Lösch [Math. Z. **34** (1931), 281–290] involving ordinary (Pringsheim) convergence.

R. P. Agnew (Ithaca, N.Y.).

Berekašvili, V. A. On Euler methods of summation of double series. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 337–342. (Russian)

Two triangular matrices a_{nk} and b_{nk} determine a transformation

$$(*)\qquad \sigma_{m,n}=\sum_{j=0}^{m}\sum_{k=0}^{n} a_{mj}b_{nk}s_{jk}$$

by means of which a double series $\sum u_{jk}$ with partial sums s_{jk} is evaluable to S if $\sigma_{m,n}\to S$ as $m,n\to\infty$. When

$$a_{nk}=b_{nk}=(q+1)^{-n}\binom{n}{k}q^{n-k},$$

(*) becomes the Euler transformation $E_{q,q}$. For the case in which $q=1$, G. N. Wollan [Proc. Amer. Math. Soc. **4** (1953), 583–587; MR **15**, 26] purported to prove that if $\sigma_{mn}\to S$ and $\sum u_{jk}$ satisfies the Tauberian conditions $|s_{jk}|\leq M$ and

$$\lim_{m,n\to\infty}(m^{1/2}+n^{1/2})(mn)^{1/2}u_{mn}=0,$$

then $\sum u_{jk}$ must converge to S. That this Tauberian statement is false is shown very simply by means of the special series $\sum u_{jk}$ for which $u_{0,k}=(-1)^k$ and $u_{jk}=0$ when $j>0$. After this observation, the author proves the following. If $q>0$, $\sigma_{mn}\to S$, and $\sum u_{jk}$ satisfies the Tauberian condition $|a_{mn}|\leq C/(m^2+n^2)$, then the series $\sum u_{jk}$ must be restrictedly convergent in the sense that if $\lambda>0$ and m and n become infinite subject to the restriction $\lambda^{-1}<m/n<\lambda$, then $s_{mn}\to S$. If $q=1$ and $\sigma>0$, the above condition involving $\sum u_{jk}$ can be replaced by

$$|u_{m,n}|\leq C(m^{(3/2)+\sigma}+n^{(3/2)+\sigma}).$$

An analogous result involving Borel exponential transforms is given: if

$$\lim_{x,y\to\infty} e^{-x-y}\sum_{j,k=0}^{\infty}\frac{x^j}{j!}\frac{x^k}{k!}u_{jk}=S$$

and $|u_{m,n}|\leq C(m^2+n^2)$, then $\sum u_{jk}$ converges restrictedly to S. The author does not say and the reviewer does not know whether the hypotheses are strong enough to imply convergence as well as restricted convergence of $\sum u_{jk}$.

R. P. Agnew (Ithaca, N.Y.).

Širokov, F. V. On Mercer's theorem. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 167–170. (Russian)

Linear functionals are used to give a direct proof of the following Mercerian theorem of E. R. Love [J. London Math. Soc. **27** (1952), 413–428; MR **14**, 159]. If

$$0<L=\limsup_{n\to\infty}\sum_{k=1}^{\infty}|a_{nk}|<\infty,$$

if $\lim_{n\to\infty}a_{nk}=0$ for each k, if $|q|<L^{-1}$, and if $s_1, s_2, \cdots$ is a bounded sequence for which

$$\lim_{n\to\infty}[s_n-q\sum_{k=1}^{\infty}a_{nk}s_k]=0,$$

then $s_n\to 0$ as $n\to\infty$. The author asks whether a more general Mercerian theorem of the reviewer [ibid. **29** (1954), 123–125; MR **15**, 305] can be proved by similar methods. An affirmative answer has been given by Mazur and Orlicz [Studia Math. **14** (1954), 129–160; MR **16**, 814].

R. P. Agnew (Ithaca, N.Y.).

Schoonmaker, N. James. Inclusion relations among some methods of summability. Proc. Amer. Math. Soc. **7** (1956), 102–108.

The paper is concerned with inclusion relations among three matrix methods of summability, viz, the Euler methods $E(p)$, the Taylor methods $T(r)$, and certain methods related to $T(r)$ and denoted by $S(q)$. The matrices which define these methods are

$$E(p)\equiv E_{n,k}(p)=\binom{n}{k}p^k(1-p)^{n-k},\ p\neq 0;$$

$$T(r)\equiv T_{n,k}(r)=(1-r)^{n+1}\binom{k}{n}r^{k-n},\ r\neq 1;$$

$$S(q)\equiv S_{n,k}(q)=(1-q)^{n+1}\binom{n+k}{k}q^k,\ q\neq 0,\ q\neq 1.$$

The following results are proved. (1) If $|q_1|<1$, $|q_2|<1$, and $q_1\neq q_2$, there are no values of q_1 and q_2 for which $S(q_2)\supset S(q_1)$. (2) If $|p|<1$, $|q|<1$, and $|q(1-1/p)|<1$, then

$$S(q)E(1/p)=S\left(\frac{q}{p+q-pq}\right).$$

(3) If $|p|<1$, $|q|<1$, and if p and q satisfy the conditions

$$\left|\frac{1}{p}+\frac{1}{q}-1\right|=\left|\frac{1}{p}\right|+\left|\frac{1}{q}-1\right|,\ \left|\frac{1}{q}\right|>\left|\frac{1}{p}\right|+\left|1-\frac{1}{p}\right|,$$

then $S(q)\supset E(p)$. As a corollary, if $0<p<1$, $0<q<1$, and $1+1/q>2/p$, then $S(q)\supset E(p)$. (4) If $|p|<1$, $|q|<1$, and $S(q)\supset E(p)$, then

$$\left|\frac{1}{p}+\frac{1}{q}-1\right|=\left|\frac{1}{p}\right|+\left|\frac{1}{q}-1\right|,\ \text{and}\ \left|\frac{1}{q}\right|\geq\left|\frac{1}{p}\right|+\left|1-\frac{1}{p}\right|.$$

(5) If $p=r/(r-1)$ and $0<p<1$, then $E(p)\supset T(r)$. (6) If $p=r/(r-1)$ and $0<|r|<1$, then $E(p)\supset T(r)$ if, and only if, $0<p<1/2$. (7) If $0<q<1/3$, then $S(q)\supset T(q)$. The author mentions a paper by P. Vermes [Amer. J. Math. **71** (1949), 541–562; MR **10**, 699] in the bibliography at the end of the paper, but says nothing about it in the text, although it is very relevant to his remark concerning Meyer-König in the last three lines of p. 102.

R. G. Cooke (London).

Slepenčuk, K. M. Summation of some classes of divergent products. Dnepropetrov. Gos. Univ. Nauč. Zap. **41** (1953), 169–174. (Russian)

Let A be a matrix a_{nk} for which $\lim_{n\to\infty}a_{nk}=a_k$ for each $k=1, 2, \cdots$. An infinite product $\prod(1+u_k)$ for which $1+\alpha_k u_k\neq 0$ for each k is said to be evaluable (or summable) A to P if $P\neq 0$ and the product in

$$P_n=\prod_{k=1}^{\infty}(1+a_{nk}u_k)$$

converges for each n and $P_n\to P$ as $n\to\infty$. Under the above conditions, each infinite product $\prod(1+u_k)$ for which $\lim u_k=0$ is evaluable A if and only if there is a constant M such that

$$\sum_{k=1}^{\infty}|a_{nk}|\leqq M$$

for each $n=1, 2, \cdots$. When all these conditions hold, $\prod(1+u_k)$ is evaluable A to $\prod(1+a_ku_k)$. When the condition $\lim u_n=0$ is replaced by the condition $|u_n|<M$, the condition on the matrix A is replaced by the requirement that the series $\sum_{k=1}^{\infty}|a_{nk}|$ converge uniformly over the set $n=1, 2, 3, \cdots$. *R. P. Agnew* (Ithaca, N.Y.).

Serebryakov, B. G. Generalization of some summability theorems. Ukrain. Mat. Ž. **4** (1952), 204–211. (Russian)

Let $\sum a_n$ be a real series with partial sums s_n. Let a_n^+ be the maximum of a_n and 0, and let a_n^- be the minimum of a_n and 0. If $\sum a_n$ satisfies both of the conditions $na_n^+=o(1)$ and $na_n^-=O(1)$ or both of the conditions $na_n^+=O(1)$ and $na_n^-=o(1)$, then

$$\liminf_{n\to\infty} s_n=\liminf_{x\to 1-} A(x)\leqq\limsup_{x\to 1-} A(x)\leqq\limsup_{n\to\infty} s_n,$$

where $A(x)$ is the Abel transform $\sum a_kx^k$ of $\sum a_k$. This generalizes the similar theorem in which $\sum a_n$ is assumed to satisfy the condition $na_n=o(1)$. *R. P. Agnew.*

Kil'berg, E. M. On strong summability of double sequences. Dnepropetrov. Gos. Univ. Nauč. Zap. **41** (1953), 153–163. (Russian)

Let $p>0$ and let A be a matrix $a_{m,n,j,k}$ with elements defined when $1\leqq j\leqq m<\infty$ and $1\leqq k\leqq n<\infty$. A double sequence is said to be strongly evaluable A to S if

$$\lim_{m,n\to\infty}\sum_{j,k=1}^{m,n} a_{m,n,j,k}|S_{jk}-S|^p=0.$$

In trivial cases (as when the elements of A are all zero), a bounded double sequence can be convergent to one value and strongly evaluable A to a different value. The author gives conditions on A which preclude this situation, and solves numerous similar problems. *R. P. Agnew.*

Čelidze, V. G. On $C_{\alpha\beta}$ and A-integrability of functions of two variables. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 65–76. (Russian)

Let

$$F_{\alpha,\beta}(x,y)=\int_0^x dt\int_0^y\left(1-\frac{t}{x}\right)^{\alpha}\left(1-\frac{u}{y}\right)^{\beta}f(t,u)du$$

so that $f(t, u)$ is integrable $C(\alpha, \beta)$ over the quadrant $t, u\geqq 0$ with the value $I(\alpha, \beta)$ if $F_{\alpha,\beta}(x, y)\to I(\alpha, \beta)$ as $x, y\to\infty$, and $f(t, u)$ is restrictedly integrable $C(\alpha, \beta)$ over the quadrant with the value $I^*(\alpha, \beta)$ if $F_{\alpha,\beta}(x, y)\to I^*(\alpha, \beta)$ as $x, y\to\infty$ subject to special restrictions. Supposing that $-1<\alpha<\alpha'$, $-1<\beta<\beta'$, and $f(t, u)$ is integrable $C(\alpha, \beta)$ over the quadrant, the author gives a collection of conditions involving α, α', β, β' and $F_{\alpha,\beta}(x, y)$ which are sufficient to ensure that $f(t, u)$ is also restrictedly integrable $C(\alpha', \beta')$ over the quadrant and that $I^*(\alpha', \beta')=I(\alpha, \beta)$. There are more such results involving unrestricted and restricted Cesàro and Abel integrals.

R. P. Agnew (Ithaca, N.Y.).

Slepenčuk, K. M. On an analogue of Abel's theorem for infinite products. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 19–21. (Russian)

It is known that convergence of $\prod(1+u_n)$ to Q does not imply that

$$(*)\qquad \lim_{x\to 1}\prod_{n=1}^{\infty}(1+u_nx^{k_n})=Q$$

when $k_n=n$, but does imply the conclusion if k_n increases sufficiently rapidly. This paper sharpens known results by proving the following. Let $p>1$. In order that a sequence $k_1, k_2, k_3, \cdots$ of exponents be such that (*) holds whenever $\prod(1+u_n)$ converges to Q and $\sum|u_n|^{2p}<\infty$, the condition

$$(k_1^q+k_2^q+\cdots+k_n^q)/k_n^q<M$$

is necessary if $p^{-1}+q^{-1}=1$ and is sufficient if $(2p)^{-1}+q^{-1}=1$. *R. P. Agnew* (Ithaca, N.Y.)

Lotockiĭ, A. V. On a method of representing real numbers as infinite products. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. **1** (1941), no. 1, 27–35. (Russian)

Let α_0 be a real irrational number for which $1<\alpha_0<2$. A representation

$$(*)\qquad \alpha_0=\left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\left(1+\frac{1}{x_3}\cdots\right)$$

of α_0 in the form of an infinite product, in which $x_1, x_2, \cdots$ are positive integers for which $x_1>1$ and $x_{n+1}\geqq x_n^2$, is defined as follows. Let x_1 be the least integer for which $1+x_1^{-1}\leqq\alpha_0$, let α_1 be defined by $\alpha_0=(1+x_1^{-1})\alpha_1$, let x_2 be the least integer for which $(1+x_2^{-1})\leqq\alpha_1$, let α_2 be defined by $\alpha_1=(1+x_2^{-1})\alpha_2$, and so on. These representations, and similar ones in which α_0 is rational and the product may be finite, are studied. The products converge extremely rapidly, and are particularly interesting when α is the square root of a rational number. In case q is a positive integer, $p=q+2$, and $\alpha_0=(p/q)^{\frac{1}{2}}$, (*) holds when $x_1=q+1$, $x_2=2x_1^2-1$, $x_3=2x_2^2-1$, $\cdots$. For example,

$$3^{\frac{1}{2}}=\left(1+\frac{1}{2}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{97}\right)\left(1+\frac{1}{18817}\right)\cdots,$$

the denominators being 2, $2\cdot 2^2-1$, $2\cdot 7^2-1$, $2\cdot 97^2-1$, $\cdots$. Use of 12 factors would give $3^{\frac{1}{2}}$ correct to 1000 decimals. In case p and q are positive integers for which $p\neq q+2$ and $(pq)^{\frac{1}{2}}$ is irrational, formulas for $(p/q)^{\frac{1}{2}}$ and $(pq)^{\frac{1}{2}}$ are obtained by selecting suitable values of x and y satisfying the Pell equation $x^2-pqy^2=1$ and using the identity

$$\left(\frac{p}{q}\right)^{\frac{1}{2}}=\frac{py}{x}\left(\frac{2x^2}{2pqy^2}\right)^{\frac{1}{2}}=\frac{py}{x}\left(\frac{2pqy^2+2}{2pqy^2}\right)^{\frac{1}{2}}$$

and the result of multiplying this by q. For example,

$$21^{\frac{1}{2}}=\frac{9}{2}\left(1+\frac{1}{55}\right)\left(1+\frac{1}{2\cdot 55^2-1}\right)\cdots.$$

[For a similar discussion of infinite products and for historical remarks, see Sierpiński, Bull. Soc. Roy. Sci. Liège **22** (1953), 520–529 [MR **15**, 699]; the paper under review seems to have been published 12 years before that of Sierpiński.] *R. P. Agnew* (Ithaca, N.Y.).

Kalašnikov, M. D. Theorems of Tauberian type for infinite products. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 318–322. (Ukrainian. Russian summary)

Let A be a matrix a_{nk} for which $\lim_{n\to\infty}a_{nk}=a_k$ for each $n=1, 2, \cdots$. Let $k_1, k_2, \cdots$ and M be such that

$$\sum_{k=1}^{k_n}|a_{nk}-1|+\sum_{k=k_n+1}^{\infty}|a_{nk}|<M.$$

Let $0<\alpha\leqq 1$. In order that $\prod(1+u_k)$ be convergent whenever

$$\lim_{n\to\infty}\prod_{k=1}^{\infty}(1+a_{nk}u_k)=P\neq 0$$

and u_k satisfies the Tauberian condition $u_k=O(k^{-\alpha})$ it is necessary and sufficient that $k_{n+1}-k_n=O(k_n^{\alpha})$.

R. P. Agnew (Ithaca, N.Y.).

See also: Ionescu, p. 830; Nicolescu, p. 838.

Interpolation, Approximation, Orthogonal Functions

Havinson, S. Ya. On uniqueness of a polynomial of best approximation in the metric of the space L_1. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1159–1161. (Russian)

Let μ be a measure on a σ-ring of subsets, containing all Borel sets, of a separable metric space R, and let E be a μ-measurable subset of R. For each $p\in\bar{E}$ and each complex-valued μ-measurable f on E, set

$$A(p,E,f)=\bigcap_{r>0}\overline{f(E\cap[N(p;r)-\{p\}])}.$$

Then there exists $E_1\subset E$ such that $\mu(E-E_1)=0$ and for any other $E_2\subset E_1$, with $\mu(E_1-E_2)=0$, $A(p,E_1,f)=A(p,E_1,f|E_2)$ for each $p\in\bar{E}_1$. If, in addition, $f\in L_1(E)$ (the class of μ-integrable functions on E) and if, for each $p\in\bar{E}_1$, $A(p,E_1,f)$ contains non-collinear points or a (perhaps degenerate) non-empty linear segment, then f is said to be of class T. The author's main result is: If H is a linear manifold of continuous, complex-valued functions, whose L_1-closure is not all of $L_1(E)$, then the best L_1-approximation (when it exists) is unique for every $\lambda\in T$ if and only if there exists no closed $F\subset E$ with the properties (a) $\mu(E-F)>0$, (b) there exists $\phi_0\in H$ such that $\phi_0(x)=0$ precisely on F, and (c) there exists α such that $|\alpha(x)|\leqq 1$ almost everywhere on E, $\alpha(x)=\pm|\phi(x)|/\phi(x)$ on $E-F$ with one sign or the other on each component of $E-F$, and $\int_E\alpha\phi\,d\mu=0$ for $\phi\in H$. Six examples are cited as applications of this result. One of these (E is a region of the complex number plane, μ is a Borel measure assigning positive measure to each subregion of E, H is a class of functions analytic on E, and $\lambda\in T$) shows that the best approximation may be unique even though H is generated by a non-Tchebicheff system (the continuous functions $u_1,\cdots,u_n$ on an interval E form a Tchebicheff system if $\sum_1^n\alpha_iu_i(x)=0$ for n distinct x's implies that $\alpha_1=\cdots=\alpha_n=0$). Finally, the following result is quoted: If $E\subset(-\infty,\infty)$, μ is Lebesgue measure, and H is a linear manifold of $l_1(E)$ (the class of all real-valued μ-integrable functions on E) for which $\bar{H}=l_1(E)$, then the class of λ's for which the best l_1-approximation is not unique forms a set everywhere dense in $l_1(E)$. *A. E. Livingston.*

Sapogov, N. A. On best approximation of analytic functions of several variables and on series of polynomials. Mat. Sb. N.S. **38(80)** (1956), 331–336. (Russian)

Let $E_n(f)$ be the best approximation by polynomials of degree $\leqq n$ to a function $f(x)$ on E: $|x|\leqq 1$. S. Bernstein proved that $f(x)$ is analytic in the ellipse C: $|z+(z^2-1)^{\frac{1}{2}}|<R$, and in no larger ellipse, if and only if $\limsup(E_n(f))^{1/n}=1/R$. This result (and some closely related ones) are generalized to functions of k complex variables; E is replaced by $E\times E\times\cdots\times E=E^k$, C by $C\times C\times\cdots\times C=C^k$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Voronovskaya, E. V. Application of functional analysis to polynomials of least deviation. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 5–8. (Russian)

A set of real numbers $\mu_0,\cdots,\mu_n$ defines a linear functional F in the space of polynomials of degree $\leqq n$:

$$(px)=\sum_{k=0}^{n}c_kx^k:\ F(p)=\sum_{k=0}^{n}\mu_kc_k.$$

If the polynomials $p(x)$ are considered as elements of the Banach space $C=C(0,1)$, then F can be extended to a functional on C without any increase in the norm $\|F\|$. Like any functional on C, F has the representation $F(f)=\int_0^1 f(x)dg(x)$. The special nature of F entails that there is a polynomial $p(x)$ such that (1) $\|p\|=1$, $F(p)=\|F\|$. This is only possible if $g(x)$ is a step function and if $p(x)=\pm 1$ at every discontinuity of $g(x)$. The author investigates the properties of extremal polynomials for which (1) holds. Typical results are the following. 1. If $\mu_0,\cdots,\mu_n$ are not the first $n+1$ moments of a monotone function $\phi(t)$ ($\mu_k=\int_0^1 t^k d\phi(t)$), then the sequence $\mu_0,\cdots,\mu_n$ can be completed in one and only one way to an infinite sequence of moment constants $\mu_k=\int_0^1 t^k dg(t)$, $\int_0^1|dg|<\infty$, such that, for every $p\geqq n$, $\{\mu_0,\cdots,\mu_p\}$ defines a functional F_p with $\|F_p\|=\|F\|$. 2. If $\mu_0,\cdots,\mu_n$ are not the first $n+1$ moments of a monotone function, then there is a unique integer s and a unique set of numbers $\sigma_1,\cdots,\sigma_s,\delta_1,\cdots,\delta_s$ such that

$$\mu_k=\sum_{j=1}^{s}\delta_j\sigma_j^k\quad(0\leqq\sigma_1<\sigma_2<\cdots<\sigma_s\leqq 1;\ k=0,\cdots,n).$$

3. If $s>\frac{1}{2}n+1$, the extremal polynomial is unique. 4. For given $\mu_0,\cdots,\mu_{k-1},\mu_{k+1},\cdots,\mu_n$ there is an interval (μ_k',μ_k'') such that for μ_k outside the interval the only extremal function is

$$T_n(x)=\pm\cos(n\arccos(2x-1));$$

for μ_k inside the interval T_n is never an extremal function. 5. Every extremal polynomial which takes the absolute value one at $s>\frac{1}{2}n+1$ points and which has opposite sign at consecutive such points is the extremal polynomial of a functional defined by $\mu_k=0$ $(k<s-1)$, $\mu_{s-1}=\pm 1$, $\mu_s,\cdots,\mu_n$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Voronovskaya, E. V. Extremal polynomials of some of the simplest functionals. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 193–196. (Russian)

This paper is a continuation of the one reviewed above. It gives a detailed investigation of extremal polynomials for which $s=n$. All polynomials which can be extremal polynomials of this type are characterized. They arise from the functionals defined by $\mu_k=0$ $(k<n-1)$, $\mu_{n-1}=-1$, $\mu_n=\theta$ with $-\frac{1}{2}(n+1)<\theta<-\frac{1}{2}(n-1)$. Also characterized are extremal polynomials p taking on the absolute value one at n points $\xi_1,\cdots,\xi_n$ and such that $p(\xi_j)p(\xi_{j+1})=1$ for exactly one index j. *W. H. J. Fuchs.*

Rymarenko, B. A. On a problem analogous to the problems of E. I. Zolotarev - N. I. Ahiezer. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 129–133. (Russian)

The problem is to find the smallest variation on $(-1,1)$ of a polynomial of degree n which is nondecreasing on the real axis and has its three highest coefficients assigned. [The solution was announced without proof in Dokl. Akad. Nauk SSSR (N.S.) **71** (1950), 1029–1032; MR **11**, 662.] The problem reduces to finding the minimum of $\int_{-1}^{1}y_n'(x)dx$, where y_n is a polynomial of degree n with

$y_n(-1)=0$, the three highest coefficients of y_n are assigned, and $y_n'(x)\geqq 0$. The last condition implies that y_n' can be written as the sum of two squares of polynomials; these polynomials are expanded in normalized Legendre polynomials; the Parseval relation reduces the problem to an algebraic minimum problem which can be solved by elementary means. *R. P. Boas, Jr.*

Ezrohi, T. G. Treatment by functional analysis of a general method of constructing remainder terms for various linear formulas of multidimensional approximate analysis. Kiĭv. Derž. Ped. Inst. Nauk. Zap. 16, Fiz.-Mat. Ser. no. 5 (1954), 41–87. (Ukrainian)

E. J. Rémès has published a general method for the determination of the remainder terms in formulas for the approximation of linear functionals in spaces of functions of a single real variable [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 28 (1940), 396–399; Acad. Sci. RSS Ukraine. Rec. Trav. [Zb. Prace] Inst. Math. 1940, no. 4, 47–82; MR 2, 194, 195]. The present work is an extension of the above mentioned method to the determination of the remainder terms of approximations to linear functionals in spaces of functions of several variables. *H. P. Thielman.*

Šilov, G. E. On some criteria of closure and completeness of a system of functions. Kiĭv. Derž. Univ. Nauk. Zap. 12 (1953), Mat. Sb. no. 7, 37–48. (Russian)

The author defines the set $S=\{\phi_n(t)\}\subset C(a, b)$ as closed, if $g\in C(a, b)$, $\phi\in S$, implies $\int_a^b g(t)\phi(t)dt=0$. An example is given to show the simple fact that a set may be closed, but not fundamental (in the sense of Banach) in $C(a, b)$. The rest of the paper contains standard proofs of the fact that polynomials and trigonometric polynomials are fundamental in $C(a, b)$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Grebenyuk, D. G. Some theorems relative to polynomials of several variables deviating least from a given function whose coefficients are connected by several linear relationships. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 10 (1953), no. 2, 105–128. (Russian)

Let (1) $p(x, y)=\sum p_{r_1,r_2}x^{r_1}y^{r_2}$ be a set of polynomials, coefficients of which satisfy the equations

$$\sum \alpha_{r_1,r_2}p_{r_1,r_2}=\varepsilon_1,\quad \sum \beta_{r_1,r_2}p_{r_1,r_2}=\varepsilon_2,$$

where $r_1+r_2=0, 1, \cdots, n$, and ε_1 and ε_2 equal zero or one, but are not both zero at the same time, and let $\phi(x, y)$ be a one-valued continuous function in a given plane domain. The author presents a number of theorems, concerning the subsets of polynomials (1) deviating least from $\phi(x, y)$, analogous to those in the theory of polynomials of one variable deviating least from zero. *S. Kulik.*

Batyrev, A. V. On the possibility of approximating continuous functions by rational ones. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 99–104. (Russian)

Let E be a bounded, closed, nowhere dense set on the z-plane. Let $\{M\}$ be the collection of all simply connected, open sets containing ∞ bounded by a Jordan curve and such that their complement lies in the intersection of CE with the disc $|z-\zeta|<\delta$, where ζ is a fixed point of E. Let $t=\phi(z)$ be the mapping function of M on $|t|>a(M)$, where ϕ is normalized by $\phi(\infty)=\infty$, $\phi'(\infty)=1$. Define $a(\delta, \zeta)=\sup_M a(M)$, $a(\delta)=\inf_{\zeta\in E} a(\delta, \zeta)$. The author proves that $\delta^2=o(a(\delta))$ as $\delta\to 0$ is a sufficient condition for the uniform approximability on E of a continuous function by rational functions; $\delta^2=O(a(\delta))$ is a necessary condition. The work follows closely the method of S. Mergelyan [Uspehi Mat. Nauk (N.S.) 7 (1952), no. 2 (48), 31–122 MR 14, 547; 15 612]. *W. H. J. Fuchs* (Ithaca, N.Y.).

Tavadyan, A. B. On completeness of polynomials in several variables for weighted mean-square approximation. Akad. Nauk Armyan. SSR. Dokl. 21 (1955), 149–156. (Russian. Armenian summary)

A theorem on completeness of polynomials in the space L_w^2 with norm

$$\|f(x_1\cdots x_k)\|^2 = \int_{-\infty}^{\infty}\cdots\int e^{-q_1(x_1)-\cdots-q_k(x_k)}|f(x_1\cdots x_k)|^2dx_1\cdots dx_k.$$

The theorem is an almost immediate consequence of the corresponding one-dimensional theorems which may be found in the Russian literature [see Džrbašyan, Mat. Sb. N.S. 36(78) (1955), 353–440; MR 17, 31]. *W. H. J. Fuchs* (Ithaca, N.Y.).

Geronimus, Ya. L. On a simple method of constructing biorthogonal systems. Ukrain. Mat. Ž. 7 (1955), 267–272. (Russian)

Let f be a linear functional in a space of complex-valued functions. Starting from a set of elements $P_n(z)$ satisfying $f(P_n\bar{P}_m)=\delta_{nm}$, the author constructs sets of functions $(u_r(z), v_s(z))$ satisfying $f(u_r\bar{v}_s)=\delta_{rs}$ by the following (formal) method. Let $A=(a_{kn})_{k,n=0}^{\infty}$ be an infinite matrix with real elements, $a_{kn}=0$ $(k>n)$. Let P be the column vector with components P_n $(n=0, 1, \cdots)$, V the column vector with components $v_0, v_1, \cdots$, U the column vector with components $u_0, u_1, \cdots$. Define $V=AP$; $B=A^{-1}$ $(b_{kn}=0$ for $k>n)$, $U=B^{tr}P$. Several special cases are given. Typical is the system

$$f(w)=\frac{2}{\pi}\int_0^{2\pi} w(\theta)d\theta,$$

$$v_s(\theta)=-\tfrac{1}{2}\log\left|\tan\left(\frac{\pi}{4}-\frac{s\theta}{2}\right)\right|,$$

$$u_r(\theta)=\sum_{(2m+1)|r}\frac{(-1)^m\mu(2m+1)}{2m+1}\sin\frac{r\theta}{2m+1},$$

where $\mu(x)$ is the Möbius function. *W. H. J. Fuchs.*

Alexits, G. Eine Bemerkung zur starken Summierbarkeit der Orthogonalreihen. Acta Sci. Math. Szeged 16 (1955), 127–129.

Let $\{\varphi_n\}$ be an orthonormal system on the finite interval (a, b), and let $\sigma_n^{\alpha}(x)$ denote the nth (C, α) means of the series (*) $\sum_{n=1}^{\infty} c_n\varphi_n(x)$. The author shows that the Abel summability almost everywhere (a.e.) (a, b) of the series (*) to a function $s(x)$ implies for $\alpha>\frac{1}{2}$ that

$$(**)\qquad \sum_{m=1}^{n}[s(x)-\sigma_{\nu_m}^{\alpha-1}(x)]^2=o(n)$$

a.e. on (a, b), for every increasing subsequence $\{\nu_m\}$ of positive integers provided $c_n^2=O(n^{-1}\lambda_n^{-1})$ for some increasing sequence $\{\lambda_n\}$ of positive numbers for which $\sum(n\lambda_n)^{-1}<+\infty$ and $\{n/\lambda_n\}$ is increasing. The proof follows that of Zygmund's theorem in the case $\nu_m=m$ where (**) obtains under the sole condition that $\sum c_n^2<+\infty$. *G. Klein* (Cambridge, Mass.).

Orts, J. M^a. Contribution to the problem of analytic continuation of Legendre series. Collect. Math. 7 (1954), 97–112. (Spanish)

The author is concerned with transforming a Legendre

series $\sum_0^\infty a_n P_n(x)$, convergent inside some ellipse, by means of the substitution $x=(1-y)/(1+y)$ and then expressing the resulting series as $\sum_0^\infty a_n' P_n(y)$ or as this series multiplied by an appropriate function of y, a procedure somewhat like the Euler transformation of a power-series. He indicates two schemes for determining $(a_n')_0^\infty$: (1) use the formula

$$\sum_0^\infty a_n P_n(x) = -i\pi^{-1}\int_{x-i\sqrt{(1-x^2)}}^{x+i\sqrt{(1-x^2)}} \frac{\sum_0^\infty a_n z^n}{(1-2xz+z^2)}\,dz$$

[E. T. Whittaker and G. N. Watson, Modern analysis, 4th ed., Cambridge, 1927, p. 315], and make the transformation $(x, z)\to((1-y)/(1+y), u)$, where $(1+z)^2/(1-z)^2 = -2u/(1+u^2)$; (2) use the identity

$$P_n((1-y)/(1+y)) = (1+y)^{-n}\sum_{k=0}^{n}\binom{n}{k}^2(-y)^k$$

[G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Bd. II, Springer, Berlin, 1925, pp. 92, 288]. In order to use these schemes, he needs to determine a sequence $(c_n)_0^\infty$ so that $\sum c_n x^n = \sum a_n P_n(x)$ inside the minor circle of the ellipes of convergence of the Legendre series and, conversely, assuming that a power-series $\sum c_n' x^n$ can be continued analytically to the interior of a nondegenerate ellipse with foci at $x=\pm 1$, to find $(a_n')_0^\infty$ so that $\sum a_n' P_n(x) = \sum c_n' x^n$ inside the circle of convergence of the power-series. He also indicates how to determine $(b_n)_0^\infty$ so that $\sum_0^\infty B_n T_n(x) = \sum b_n P_n(x)$ in a common ellipse of convergence, T_n being the Tchebichef polynomial of degree n. *A. E. Livingston.*

Campbell, Robert. Détermination de la famille des polynomes orthogonaux dont les dérivés sont orthogonaux. C. R. Acad. Sci. Paris **242** (1956), 1110–1111.

Let $\{P_n(x)\}$ be a system of orthogonal polynomials, with the leading coefficients one, satisfying the recurrence relation $P_n=(x+r_n)P_{n-1}-C_nP_{n-2}$ and let $P_n' = n(n-1)^{-1}(x+\varrho_n)P_{n-1}'-\gamma_nP_{n-2}'$, where $P_n\equiv P_n(x)$, $P_n'\equiv P_n'(x)$, and C_n and γ_n are positive. The author derives relations between the coefficients r_n, C_n, ϱ_n, and γ_n in order to make the system $\{P_n'(x)\}$ orthogonal also. *S. Kulik* (Columbia, S.C.).

Badalyan, G. V. Generalization of Legendre polynomials and some of their applications. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauk **8** (1955), no. 5, 1–28. (Russian. Armenian summary)

Consider a nondecreasing sequence of real numbers γ_n (greater than $-\frac{1}{2}$) and associate with γ_n the function $\omega_n(x)$ which is x^{γ_n} if $\gamma_{n-1}<\gamma_n<\gamma_{n+1}$; if k consecutive γ_n are equal, the corresponding $\omega_n(x)$ are x^{γ_n}, $x^{\gamma_n}\log x$, $x^{\gamma_n}(\log x)^2$, $\cdots$, $x^{\gamma_n}(\log x)^{k-1}$. The author is interested in generalizing various concepts and theorems, first from $\{x^n\}$ to $\{x^{\gamma_n}\}$ when the γ_n are distinct, and then to $\omega_n(x)$ when the γ_n are not necessarily distinct. The generalized Legendre polynomials are obtained by orthonormalizing x^{γ_n} on (0, 1) with unit weight-function, and are exhibited in the form of contour integrals. The author also considers the corresponding orthonormal system on $(0, \infty)$ with weight-function e^{-x}. A system $\{\omega_n(x)\}$ is L^2-complete if $\sum 1/(\gamma_n+1)=\infty$ [Badalyan, Akad. Nauk Armyan. SSR. Soobšč. Inst. Mat. Meh. **5** (1950), 13–84; MR **15**, 946; Gelfond, Calculus of finite differences, Moscow-Leningrad, 1952, p. 117; MR **14**, 759]; the author now shows that this condition is also necessary. He discusses various moment problems for the $\omega_n(x)$. *R. P. Boas, Jr.*

Shenton, L. R. A determinantal expansion for a class of definite integral. III. Generalised continued fractions. Proc. Edinburgh Math. Soc. (2) **10** (1956), 134–140.

The expansion obtained in part I [same Proc. **9** (1953), 44–52; MR **15**, 781] is specialized to obtain a determinantal expansion of

$$\int_b^a \frac{w(x)dx}{P(x)},$$

where P is a polynomial. Several forms of this expansion are investigated and their relation to the continued fraction in case of a linear polynomial $P(x)$ is investigated. *A. Erdélyi* (Pasadena, Calif.).

See also: Erwe, p. 835.

Trigonometric Series and Integrals

★ **Zygmund, Antoni. Trigonometrical series. 2nd ed.** Chelsea Publishing Co., New York, 1952. vi+329 pp. Paperbound: $1.50; clothbound: $4.95.

Photo-offset reprint, with some corrections, of the first edition [Warszawa-Lwów, 1935].

Kiš, O. On convergence of trigonometric interpolation of periodic analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 449–450. (Russian)

Let $f_n(z)=\sum_{k=0}^n (a_n \operatorname{Re} z^k + b_n \operatorname{Im} z^k)$ be the harmonic polynomial which interpolates the function $f(z)$ analytic for $|z|\leq 1$ at the points $\exp ix_k^n$ $(0\leq x_k^n<2\pi,\ k=1, \cdots, 2n+1,\ n=0, 1, \cdots)$. If some of the x_k^n are repaeated, the corresponding number of derivatives of $f_n(z)$ at $\exp ix_k^n$ agree with those of $f(z)$.

Let $$\varrho(x)=\lim_{\substack{a\to x+0\\ b\to x-0}}\lim_{n\to\infty} n[a, b]/(2n+1)(b-a),$$

where $n[a, b)$ is the number of k's for which $a\leq x_k^n<b$. The principal theorems state: 1) A necessary and sufficient condition that $f_n(z)\to f(z)$ pointwise for every $f(z)$ analytic for $|z|\leq 1$ is that the limiting density $\varrho(x)$ exist and equal $(2\pi)^{-1}$ for every x $(0\leq x<2\pi)$, and if this condition is met the convergence is uniform. 2) If $f(z)$ in analytic for $|z|\leq(1+\sqrt{2})^2$, then $f_n(x)\to f(x)$ uniformly for $|z|\leq 1$ whatever the choice of the points x_k^n (the bound $(1+\sqrt{2})^2$ is best under these conditions). The results encompass the case of trigonometrical interpolation of periodic analytic functions on the interval $[0, 2\pi]$. *G. Klein.*

Izumi, Shin-ichi; and Satô, Masako. Some trigonometrical series. XVIII. Proc. Japan Acad. **32** (1956), 20–23.

Supplementary conditions are imposed which yield convergence instead of $(C, 1)$ summability in Parseval's relation. Examples of the type of conditions obtained are

$$(1)\qquad \int_0^h \{f(x+u)-f(x-u)\}\,du = o\left(h/\log(1/h)\right)$$

uniformly in x as $h\to 0$ in addition to the usual g integrable and f bounded and measurable and

$$(2)\qquad F(t)=\delta^{-1}\int_0^h \{f(t+u)-f(t-u)\}\,du$$

of total variation of order $o(1/\log(1/\delta))$ as $\delta\to 0$ in addition to the usual f of bounded variation and g continuous. *P. Civin* (Eugene, Ore.).

Salem, R. New theorems on the convergence of Fourier series. Nederl. Akad. Wetensch. Proc. Ser. A. 57= Indag. Math. **16** (1954), 550–555.

The Fourier series $S(f)$ of a function $f \in L_1(0, 2\pi)$ converges almost everywhere in a set E of positive measure, according to a well-known theorem of Marcinkiewicz, if $\int_0^h |f(x+t)-f(x-t)|dt=O(h/\log h^{-1})$ $(h\to 0+)$ on E. The author obtains the following convergence criteria based on the integral $I=\int_0^h \{f(x+t)-f(x-t)\}dt$, which include that of Dini-Lipschitz: 1) If $I=o(h/\log h^{-1})$ uniformly on a closed interval E, then $S(f)$ converges almost everywhere on E, uniformly on every open subinterval of E if f is continuous on E, and uniformly everywhere if $E=[0, 2\pi]$ as well. 2) If $I=O(h/\log h^{-1})$ uniformly for all x, then the partial sums of $S(f)$ are dominated by a function $g \in L^{1-\varepsilon}$ for every positive ε; moreover, $g \in L^r$ $(r>1)$ if $f \in L^r$, and $g \in L$ if $f \log^+|f| \in L$. *G. Klein* (Cambridge, Mass.).

Satô, Masako. Uniform convergence of Fourier series. V. Proc. Japan Acad. **31** (1955), 600–605.

Salem, in the paper reviewed above, proved that, if $h^{-1}\int_0^h (f(x+t)-f(x-t)dt=o(\log 1/h)^{-1}$ as $h\to+0$, uniformly for all x, then the Fourier series of f converges p.p., and uniformly in any closed interval contained in an open interval of continuity of f. A new proof of this theorem is given together with certain generalizations of the following type. The o-condition is relaxed (for instance, to $o(\log 1/h)^{-\alpha}$, $0<\alpha<1$) while certain order conditions on the Fourier coefficients ($O(e^{(\log n)^\alpha}/n)$ in the above case) are added [cf. Satô, Proc. Japan Acad. **30** (1954), 809–813; MR **16**, 1015]. *W. W. Rogosinski.*

Hirokawa, Hiroshi. Uniform convergence of some trigonometrical series. Tôhoku Math. J. (2) **6** (1954), 162–173; remarks and errata (2) **7** (1955), 297–298.

Suppose the sequence $\{a_r\}$ satisfies

$$\sum_n^\infty |a_r-a_{r+1}|=O(n^{-\alpha}) \quad (0<\alpha<1).$$

For $0<\beta$, let t_n^β be the nth (C, β) sum of the sequence $\{va_v\}$. Also for $0<\beta$, if $\sum_1^\infty a_v$ is convergent, let τ_n^β be nth (C, β) sum of the sequence $\{\sum_v^\infty a_j\}$. Then either $t_n^\beta=o(n^\alpha)$ or $\tau_n^{\beta-1}=o(n^{\beta\alpha-1})$ is sufficient for the uniform convergence of each of the series $\sum_1^\infty a_v \sin vx$ and $\sum_1^\infty a_v \cos vx$ in the interval $0\leqq x\leqq\pi$. *P. Civin.*

Kinukawa, Masakiti. On the convergence character of Fourier series. II. Proc. Japan Acad. **32** (1956), 24–26.

Let $s_n(t)$ denote the nth partial sum of the Fourier series of $f(t)$. The author extends his earlier result [same Proc. **31** (1955), 513–516; MR **17**, 609] by showing the uniform convergence of

$$\sum_{n=2}^\infty |s_n(x)-f(x)|^k n^{k\alpha-1}(\log n)^{-\gamma}$$

under the hypotheses (1) $0<\alpha\leqq 1$, $\gamma>1$, $k>0$ and $f \in \mathrm{Lip}\,\alpha$, or (2) $0<\alpha<1$, $\gamma=0$, $k>0$, $\delta>1/k$ and

$$|f(x+t)-f(x)|\leqq A|t|^\alpha(\log(1/|t|))^{-\delta}.$$

P. Civin (Eugene, Ore.).

Ganea, Tudor. Gibbs phenomenon in the theory of Fourier series. Gaz. Mat. Fiz. Ser. A. **8** (1956), 10–18. (Romanian)

Expository article. *R. P. Boas, Jr.* (Evanston, Ill.).

Shapiro, Victor L. Cantor-type uniqueness of multiple trigonometric integrals. Pacific J. Math. **5** (1955), 607–622.

Given the multiple trigonometric integral

$$\int_{E_n} e^{i(x,u)}c(u)\,du,$$

where $c(u)$ is integrable in every bounded set. Let Z be a closed set of capacity zero in every sphere. The author proves that if (i) the integral is spherically summable $(C, 1)$ to zero almost everywhere, (ii) the $(C, 1)$ spherical mean is bounded in E_n-R, and (iii)$c(u)/(|u|^2+1)$ is integrable in E_n, then $c(u)$ vanishes almost everywhere. This is the integral analogue of the result obtained by the author for trigonometric series [Trans. Amer. Math. Soc. **77** (1954), 322–339; MR **16**, 242]. *S. Izumi* (Sapporo).

Methée, Pierre-Denis. Transformées de Fourier de distributions invariantes. II. C. R. Acad. Sci. Paris **241** (1955), 684–686.

En complément à une Note précédente [mêmes C.R. **240** (1955), 1179–1181; MR **16**, 1101], l'auteur présente ici en détail le calcul des images de Fourier des distributions du type S et du type H. *J. Sebastião e Silva.*

See also: Ehrenpreis, p. 876.

Integral Transforms, Operational Calculus

Jaiswal, J. P. On Meijer transform. III. Compositio Math. **12** (1956), 284–297.

[The first two parts appeared in Math. Z. **55** (1952), 385–398; Ann. Soc. Sci. Bruxelles, Ser. I. **66** (1952), 131–151; MR **14**, 43, 640.] "In this paper we have established some more properties of the Meijer transform by utilising integral representations of the Whittaker's function, the parabolic cylinder function, and the Bessel function, and also the integrals involving these functions". [From the author's introduction.] *A. Erdélyi* (Pasadena, Calif.).

Szilvay, Frau G.; und Zergényi, E. Über ein Wärmeleitungsproblem. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 253–263 (1955). (Hungarian. Russian and German summaries)

Solution by means of the Laplace transformation of the following problem: $u=u(x, t)$, $0\leqq x\leqq L$, $t\geqq 0$; $u_t=k^2u_{xx}$, $u(x, 0)=u_0x/L$, $u(0, t)=0$, $u(L, t)=u_0-at$ for $t<u_0/\alpha$, $u(L, t)=0$ for $t\geqq u_0/a$. *A. Erdélyi* (Pasadena, Calif.).

See also: Rathie, p. 846; Colino, p. 860; Fort, p. 860; Ehrenpreis, p. 877.

Special Functions

Szász, Pál. On the geometrical interpretation of the modular group. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 1–12. (Hungarian)

The author gives an original geometrical construction for the modular dissection of the upper half-plane [cf. E. T. Copson, Theory of functions of a complex variable, Oxford, 1935, Ch. 15]. Starting with two circles touching each other and the real axis, a construction is given for two more circles to touch the first two and the real axis, and so on; the semi-circles joining the points of contact on the real axis give a network of curvilinear triangles, of which we are to consider the "altitudes". Next studied is

the interpretation of a modular substitution, put into continued-fraction form, in terms of this diagram, enabling the author to give an elementary proof that regions congruent to a fundamental region fill the half-plane. *F. V. Atkinson* (Canberra).

Germain, P. Remarks on transforms and boundary value problems. J. Rational Mech. Anal. **4** (1955), 925–941.

Using solutions to simple boundary-value problems found by Green's function technique for the equation $u_{xx}+k(x)u_{zz}=0$, the author obtains transform relations, analogous to the Fourier and Fourier-Dini integral theorems, involving the solutions of the equation $u_{xx}^*+\beta^2k(x)u^*=0$. Here $k(x)$ is either positive or negative but does not change sign. It is shown that if $S_a(x,\beta)$ is the solution of the ordinary differential equation which satisfies $S_a(a,\beta)=0$ and $dS_a(a,\beta)/dx=1$ and if $f(x)=\int_0^\beta \beta\phi(\beta)$ then

$$\phi(\beta)=\frac{2\omega(\beta)}{\pi}\int_0^\beta k(x)f(x)S_a(x,\beta)\,dx,$$

where ω is a certain fixed function. Similar results hold for the solution which is 1 for $x=a$ and whose derivative vanishes there. If $k(x)$ changes sign the corresponding results do not hold because the boundary-value problems are improperly set. Some classical transforms with Bessel functions are obtained and in addition the pair of new reciprocal formulae

$$(Ft)=\int_0^\infty \beta\Phi(\beta)[J_\beta(\beta t)J_{-\beta}(\beta t_0)-J_{-\beta}(\beta t)J_\beta(\beta t_0)]d\beta,$$

$$\varrho_\beta(\beta t_0)\Phi(\beta)=\int_{t_0}^\infty f(t)[t-\frac{1}{t}][J_\beta(\beta t)J_{-\beta}(\beta t_0)-J_{-\beta}(\beta t)J_\beta(\beta t)]dt.$$

C. S. Morawetz (New York, N.Y.).

MacRobert, T. M. Some Bessel function integrals. Proc. Glasgow Math. Assoc. **2** (1956), 183–184.

The author proves the formula

$$2\int_0^\infty E(p;a_r;q;c_s:z\,\mathrm{sech}^2 u)(\sinh u)^{2n-1}\cosh u\,du$$
$$=\Gamma(n)z^nE(p;a_r-n_s:q;c_s-n:z),$$

and deduces some results involving Bessel functions and confluent hypergeometric functions. *A. Erdélyi.*

Ragab, F. M. New integrals involving Bessel functions. Acta Math. **95** (1956), 1–8.

The author specializes the parameters in known integrals involving MacRobert's E-function and thus obtains integrals involving Bessel functions. *A. Erdélyi.*

Ragab, F. M. Integrals involving products of Bessel functions. Proc. Glasgow Math. Assoc. **2** (1956), 180–182.

The author evaluates

$$\int_0^\infty t^{k-1}K_{2m}(zt^{-1/2})K_n(t)dt$$

and a similar integral involving J in place of K.
A. Erdélyi (Pasadena, Calif.).

Ahund-zade, M. Yu. On some functions occurring in the theory of shells. Dokl. Akad. Nauk Azerbaĭdžan. SSR. **9** (1953), 385–390. (Russian. Azerbaijani summary)

The author sets

$$C_n(x)\pm iS_n(x)=(\mp ix)^{-n/2}J_n(2\sqrt{(\mp ix)}),$$

defines a number of functions in terms of C_n, S_n and auxiliary parameters, and writes down numerous recurrence formulas and other relations. *A. Erdélyi.*

Rathie, C. B. A theorem in operational calculus and some integrals involving Legendre, Bessel and E-functions. Proc. Glasgow Math. Assoc. **2** (1956), 173–179.

If $f(p)$ is the operational image of $h(x)$, the author obtains

$$\int_0^\infty \cosh\nu\theta\,\mathrm{sech}\,\theta f(p\cosh\theta)d\theta=p\int_0^\infty K_\nu(px)h(x)dx$$

and some similar relations. These relations are used to evaluate a considerable number of integrals.
A. Erdélyi (Pasadena, Calif.).

Rathie, C. B. A few infinite integrals involving E-functions. Proc. Glasgow Math. Assoc. **2** (1956), 170–172.

The author uses rules of operational calculus to evaluate some integrals involving generalized hypergeometric functions. *A. Erdélyi* (Pasadena, Calif.).

Wintner, Aurel. On the harmonic analysis of hypergeometrical functions. Math. Notae **9** (1949), 78–87 (1950).

The author integrates the trigonometric form of Legendre's equation, $(y'\sin\theta)'+\lambda y\sin\theta=0$, by a Fourier cosine series, and discusses a similar series expansion of the solutions of the hypergeometric equation. *A. Erdélyi.*

Ivanenko, V. V. On a question of the structure of Lamé functions. Kiïv. Derž. Ped. Inst. Nauk. Zap. **16**, Fiz.-Mat. Ser. no. **5** (1954), 21–39. (Ukrainian)

The author investigates the Weierstrassian form of the differential equation for Lamé polynomials [see Erdélyi et al., Higher transcendental functions, vol. III, McGraw-Hill, New York, 1955, p. 56 equation (3); MR **16**, 586]. Instead of the usual expansion in powers of Weierstrass' $\wp$-function, he expands the solution in derivatives of that function. Explicit expressions for the polynomial solutions and the characteristic equation are given for $2\leqq n\leqq 10$, both in the general case and in some particular cases ($g_2=1$, $g_3=0$; $g_2=0$, $g_3=1$). In the particular cases numerical values of the separation constant are also given. [There is no reference to the extensive literature of this subject, and it is not clear in which respect the author believes he has contributed to our knowledge of Lamé polynomials.] *A. Erdélyi* (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. XI. Expansions in series of generalized hypergeometric functions. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 70–82.

[For parts I–X see MR **14**, 469, 642, 748, 979; **15**, 422, 791, 955; **16**, 1106.] The author applies the results of earlier parts to sum

$$\sum_{r=0}^\infty \frac{(d-e)_r\Gamma(b+r)}{r!\Gamma(b+d+r)}\,{}_{p+1}F_{q+1}\left[\begin{matrix}b+r, a_1, \cdots, a_p\\ b, c_1, \cdots, c_q\end{matrix};-z\right]$$

and

$$\sum_{r=0}^\infty \frac{\Gamma(c_1+r)\Gamma(c_2+r)}{r!\Gamma(c_3+r)}\,{}_{p+1}F_q\left[\begin{matrix}-r, a_1, \cdots, a_p\\ c_1, \cdots, c_q\end{matrix};z\right]$$

in closed form. He discusses many particular cases and applications of these relations. *A. Erdélyi.*

Onoe, Morio. Modified quotients of cylinder functions. Math. Tables Aids Comput. **10** (1956), 27–28.

Henrici, Peter. On generating functions of the Jacobi polynomials. Pacific J. Math. **5** (1955), 923–931.

Utilisant le fait que l'équation aux dérivées partielles

$$(1)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\mu+1}{x}\frac{\partial u}{\partial x}+\frac{2\nu+1}{y}\frac{\partial u}{\partial y}=0$$

est séparable dans le système de coordonnées (ξ, η) défini par

$$x=\frac{\operatorname{sh}\xi}{\operatorname{ch}\xi+\cos\eta},\quad y=\frac{\sin\eta}{\operatorname{ch}\xi+\cos y}$$

l'auteur en obtient des solutions de la forme

$$\psi(\xi,\eta)=(\operatorname{ch}\xi+\cos\eta)^{\mu+\nu+1}S(\operatorname{ch}\xi)T(\cos\eta)$$

en montrant que S et T satisfont à une même équation différentielle dont il exprime facilement une solution $V_k^{\mu,\nu}$ par des fonctions de Gegenbauer. Il en déduit une solution de (1) sous la forme

$$\Phi_k^{(\mu,\nu)}(\varrho,\tau)=\tilde{\omega}^{-\mu-\nu-1}V_k^{\mu}\Big(\frac{1+\varrho}{\tilde{\omega}}\Big)V_k^{\mu}\Big(\frac{1-\varrho}{\tilde{\omega}}\Big)$$

avec $\varrho=x^2+y^2$, $\tau=(x^2-y^2)/(x^2+y^2)$, $\tilde{\omega}=(1-2\varrho\tau+\varrho^2)^{\frac{1}{2}}$. L'auteur développe alors $\Phi_k^{\mu,\nu}$ en série de polynômes de Jacobi $P_n^{\nu,\mu}$:

$$\sum_0^\infty \gamma_n a_{nk}\varrho^n P_n^{(\nu,\mu)}(\tau)$$

obtenant ainsi, pour ces polynômes, une fonction génératrice plus générale que celles qu'on connait et redonnant la fonction classique pour $k=0$. *R. Campbell* (Caen).

See also: Schläfli, p. 814; Smirnow, p. 833; Jaiswal, p. 845.

Ordinary Differential Equations

★**Èl'sgol'c, L. È. Kačestvennye metody v matematičeskom analize. [Qualitative methods in mathematical analysis.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 300 pp. 10.15 rubles.

The main purpose of this volume is to acquaint the reader with the modern qualitative theory of differential and differential-difference equations with particular reference to existence and uniqueness of solutions, periodicity, asymptotic behavior and stability. The principal contribution of the book lies in its thorough discussion of various aspects of differential-difference equations.

The first two chapters are devoted to the Morse theory of critical points with a brief application to functions of a complex variable. These chapters are independent of the remainder of the volume. In chapter three, the fixed-point method of Birkhoff-Kellogg for the establishment of the existence of solutions of functional equations is discussed, beginning with the finite-dimensional theorem of Brouwer and continuing through the result of Schauder.

The fourth chapter is devoted to the study of ordinary differential equations, following the usual lines of the qualitative theory of Lyapunov and Poincaré, critical points, periodicity, asymptotic behavior and stability. In addition, there is a brief treatment of the behavior of solutions of equations of the form $dx/dt=f(x(t), x(t-1))$, and, more generally, to the study of equations of the form $dx/dt=f(x(t), x(t-a(t)))$. The fifth chapter contains a systematic account of various qualitative aspects of the solutions of these equations, with results and methods paralleling those in the previous chapter.

The sixth chapter treats the variational theory associated with functions of the form

$$\int F(t, x(t), x(t-1), x'(t), x'(t-1))\,dt$$

and

$$\int F(t, x(t), x(t-a(t)), x'(t), x'(t-a(t)))\,dt.$$

A useful bibliography of 339 titles is appended.

R. Bellman (Santa Monica, Calif.).

Katō, Tizuko. Sur les points singuliers des équations différentielles ordinaires du premier ordre. IV. Nat. Sci. Rep. Ochanomizu Univ. **5** (1955), 175–177.

The author deletes the condition $y(\tau_0-m\omega)=0$ of his previous paper [same Rep. **5** (1954), 1–4; MR **16**, 1023].

N. Levinson (Cambridge, Mass.).

Ströher, Wolfgang. Über einen gewissen Typus von Differentialinvarianten der projektiven und der apollonischen Gruppe der Ebene. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **164** (1955), 189–216.

The authors considers a local Lie group of transformations on an open plane set O. The transformations of the local group induce transformations on the space of line elements over O and also on the space of line elements of order $r>1$ on O. The author considers real-valued functions J of several points and line elements of various orders, which are invariant under the (extended) local transformation group.

For the projective group p, q, xq, xp, yq, yp, $x(xp+yq)$, $y(xp+yq)$ and the "Apollonius" group p, q, xp, yq, x^2p, y^2q the author computes several such invariants.

L. Markus (Princeton, N.J.).

Protasov, V. I. On an infinite system of linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 218–221. (Russian)

Let $\{a_k\}$ $(k=0, 1, 2, \cdots)$ be a sequence of complex numbers, and let A be the infinite matrix with element a_{j+k} in the jth row and kth column. The author considers the initial-value problem (*) $y'=Ay$, $y(0)=c$ $(\,'=d/dx)$, where c is a given initial vector, and studies the relation between the rate of growth of the elements c_k of the initial vector c and the region of existence of analytic solutions of (*). To A is associated the function f via $f(z)=\sum_{k=0}^\infty a_k z^k$, which is assumed analytic in a circle $|z|<R$. Then to e^{zA} is associated $e^{zf(z)}=\sum_{k=0}^\infty b_k(x)z^k$. Formally a solution of (*) is given by $y(x)=e^{xA}c$, or

$$(**)\qquad y_j(x)=\sum_{k=0}^\infty b_k(x)c_{j+k}\quad (j=0, 1, 2, \cdots).$$

Let $F(z)=\sum_{k=0}^\infty |a_k|z^k$, $e^{zF(z)}=\sum_{k=0}^\infty B_k(x)z^k$. The author proves that if the series $\sum_{k=0}^\infty B_k(x)|c_{k+j}|$ is convergent for all $j=0, 1, \cdots$, and any $x=r<R$, then (**) is a solution of (*). Using this result and estimates on the rate of growth of the coefficients in the expansions of analytic functions, the author states several results. Typical are: (1) If f is a polynomial of degree s with a_s as coefficient of z^s, then for any initial vector c satisfying

$$(R|a_s|es)^{1/s}\limsup_{n\to\infty}\frac{|c_n|^{1/n}}{n^{1/s}}\leq 1,$$

the problem (*) has an analytic solution in $|x|<R$; (2) If f is analytic in a circle of radius R, then for any initial vector satisfying

$$\limsup_{n\to\infty}\frac{|c_n|^{1/n}}{R}<1,$$

there exists an entire solution of (*). Some results are stated to be extensions of those given by L. I. Kamynin [same Dokl. (N.S.) **93** (1953), 397–400; MR **16**, 251].

E. A. Coddington (Los Angeles, Calif.).

Magnus, Wilhelm. Infinite determinants associated with Hill's equation. Pacific J. Math. **5** (1955), 941–951.

Recherchant pour l'équation de Hill

$$y''+4[\omega^2+q(x)]y=0 \text{ où } q(x)=2\sum_1^\infty t_n\cos 2nx$$

des solutions périodiques (π ou 2π), y_1 (paire) ou y_2 (impaire) telles que $y_1(0)=1$, $y_1'(0)=0$ et $y_2(0)=0$, $y_2'(0)=1$, l'auteur remarque que ces solutions n'existent que si et seulement si $y_{1,2}(\pi/2)$ ou $y_{1,2}'(\pi/2)$, selon les cas, est nul. Il en déduit donc que les déterminants infinis qui servent à écrire l'équation caractéristique habituelle sont liés à ces quantités; et en fait, C et S désignant les déterminants attachés aux solutions paire ou impaire y_1 et y_2, avec l'hypothèse $\sum_1^\infty |t_n|<\infty$ ces deux relations sont

$$2\omega\sin(\pi\omega)\times C=-y_1'(\pi/2),$$

$$\frac{1}{\omega}\sin(\pi\omega)\times S=2y_2(\pi/2).$$

La même idée avait été utilisée, pour des cas plus généraux, par L. Brillouin dans un travail que l'auteur ne cite pas [Quart. Appl. Math. **6** (1948), 167–178; MR **10**, 252]. Supposant ensuite que $\sum_1^\infty n^2|t_n|<\infty$, il montre que toute solution y telle que $y(0,\omega)=0$, $y'(0,\omega)=b$ est exprimable sous la forme:

$$y(x,\omega)=a\cos 2\omega x+\int_{-x}^{+x}G(x,0)e^{2i\omega\theta}d\theta.$$

La fonction $G(x,\theta)$, définie dans $-|x|\leqq\theta\leqq+|x|$, étant telle que

$$G(x,x)=G(x,-x)=\frac{b}{2}-a\sum_1^\infty\frac{t_n}{n}\sin 2\pi x$$

et satisfaisant à l'équation:

$$\frac{\partial^2G}{\partial x^2}-\frac{\partial^2G}{\partial\theta^2}+4q(x)G=0.$$

R. Campbell (Caen).

Solncev, Yu. K. Two examples of dynamical systems determined by infinite systems of differential equations. Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952), 156–167. (Russian)

Let S be the space of countable sequences of real numbers, $\xi=(x_1,\cdots,x_n,\cdots)\in S$. Let $M\subset S$ have a "norm" $N(\xi)$ which is a real number for each $\xi\in M$ with $N(0)=0$, $N(\xi)=N(-\xi)>0$ for $\xi\neq 0$, $N(\xi_1+\xi_2)\leqq N(\xi_1)+N(\xi_2)$, N satisfies a certain type of Minkowski inequality, and suppose that the resulting metric on M is complete. Examples of such metric spaces are the Banach spaces l_p ($p\geqq 1$).

Consider the ordinary differential system $d\xi/dt=\varphi(t,\xi)$, where φ is defined on a subset of the product of M and a real interval T and φ has its range in M. The author proves the usual theorems concerning existence, uniqueness and continuity of solutions $\xi(t,t_0,\xi_0)$ with initial conditions $\xi(t_0,t_0,\xi_0)=\xi_0\in M$. The hypotheses involve continuity and boundedness of φ, a uniform Lipschitz condition on $\varphi(t,\xi)$ in ξ, and the proof is effected through the successive iterations of Picard. *L. Markus.*

Herbst, Robert Taylor. The equivalence of linear and nonlinear differential equations. Proc. Amer. Math. Soc. **7** (1956), 95–97.

Thomas [same Proc. **3** (1952), 899–903; MR **14**, 558] raised the question: What equations of order n have general solution expressible as $F(u_1,\cdots,u_n)$, where $u_1,\cdots,u_n$ constitute a variable set of solutions of a fixed linear differential equation? The author gives the complete answer to this question for linear homogeneous equations of the second order: If u, v are variable independent solutions with Wronskian w of the equation $Y''-w(x)^{-1}w'(x)Y'+q(x)Y=0$, where w and q are given functions , then the equation $y''-w^{-1}w'y'=f(y,y',w,q)$ has general solution $y=F(u,v)$ if and only if $f=-qZ(y)+A(y)y'^2+w^2C(y)$, where Z, A, C satisfy $ZC'+(3-AZ)C=0$, $Z'-AZ=1$. *M. Zlámal* (Brno).

Afriat, S. N. Simultaneous linear differential equations with constant coefficients. Proc. Cambridge Philos. Soc. **52** (1956), 209–212.

The author considers a system

$$(*)\quad\sum_k\sum_i l_{jk}^{(i)}X_k^{(i)}=F_j\quad(i=0,1,\cdots,n;\ j,k=1,2,\cdots,m)$$

of m simultaneous linear differential equations of order n with constant coefficients in m functions X_k. The F_j are analytic. If the matrix of coefficients of the nth derivatives is nonsingular, the system may be written in matrix form: $U'+CU=V$, with U a vector of order mn, C a square matrix of order mn and V a known vector of order mn. The author derives the solution of this involving the exponential $\exp(Ct)$. An expression for $\exp(Ct)$ as a polynomial in C with coefficients analytic in t is given. The characteristic equation for C is found from the coefficients of (*). *W. S. Loud* (Cambridge, Mass.).

Afriat, S. N. The approach to scalar growth of a vector transformed by an increasing power of a matrix. Proc. Cambridge Philos. Soc. **52** (1956), 213–214.

Using the expression for $\exp(Ct)$ given in the paper reviewed above, the author investigates the behavior of $\exp(Ct)Z_0$ for large t where Z_0 is a constant vector. If a is the matrix $\exp C$, α is the largest characteristic value of a (assumed real and simple) and θ is the largest characteristic value (other than unity) of a/α, the author obtains the fact that $(a/\alpha)^nZ_0=Z_\infty+O(\theta^n)$ with Z_∞ a characteristic vector of a belonging to α. *W. S. Loud.*

Atrašenok, P. V. Determination of the freedom of choice of a matrix reducing a system of linear differential equations to a system with constant coefficients. Vestnik Leningrad. Univ. **1953**, no. 2, 17–29. (Russian)

Let $\mathfrak{M}$ be the set of matrices $Z(t)$ which, on $[0,\infty)$, are of class C^1, non-singular, and bounded together with Z^{-1}. The matrix equation (1) $\dot X=XA(t)$ with A of class C and bounded on $[0,\infty)$ is reducible, i.e. it can be transformed by $Y=XZ$ with $Z\in\mathfrak{M}$ to $\dot Y=YB$ with B constant if and only if $U=Ze^{-Bt}$ is a solution of (2) $\dot U=-A(t)U$ [N. Erugin, Trudy Mat. Inst. Steklov. **13** (1946); MR **9**, 509]. The author assumes $Z=e^{-Bt}Ke^{Bt}$ with B, K constant and B in canonical form and considers conditions on K so

that Z be in $\mathfrak{M}$. It appears to the reviewer that $U=e^{-Bt}K$ is a solution of (2) and hence $A(t)=B$.

H. A. Antosiewicz (Washington, D.C.).

Vinograd, R. È. Necessary and sufficient criteria for the behavior of the solutions of a regular system. Mat. Sb. N.S. **38(80)** (1956), 23–50. (Russian)

This paper contains a detailed study of the possible modes of behavior of the functionals

$$\limsup_{t\to\infty}[\log|x_i(t)|/t],\quad \liminf_{t\to\infty}[\log|x_i(t)|/t],$$

of the solution of the linear system $dx_i/dt=a_{ij}(t)x_j$, $i=1, 2, \cdots, n$, under various assumptions concerning the coefficients $a_{ij}(t)$. *R. Bellman* (Santa Monica, Calif.).

Gambill, Robert A. Criteria for parametric instability for linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma **6** (1955), 37–43.

The author considers the linear differential system

$$\ddot{y}_j+\sigma_j^2y_j+\lambda\sum_{h=1}^{n}\varphi_{jh}(t)y_h=0\quad(j=1, 2, \cdots, n),$$

where A) $\sigma_1, \cdots, \sigma_n$ are distinct positive numbers, B) λ is a small real parameter, C) $\varphi_{jh}(t)$ are real periodic functions of period $T=2\pi/\omega$, with mean zero, and with absolutely convergent Fourier series. D) $m\omega\neq\sigma_j\pm\sigma_h$ $(j, h=1, \cdots, n;\ m=1, 2, \cdots)$.

In an earlier paper [same Riv. **5** (1954), 169–181; MR **17**, 36] the author showed that if the matrix $\Phi(t)=\|\varphi_{jh}(t)\|$ is symmetric or even, then the differential system is parametrically stable at $y=0$, $\lambda=0$. That is, given $\varepsilon>0$, there exists $\delta>0$ such that for every $|\lambda|<\delta$, $|y_i(0)|<\delta$, $|\dot{y}_i(0)|<\delta$ one has $|y_i(t)|<\varepsilon$, $|\dot{y}_i(t)|<\varepsilon$ on $t\geqq0$, $i=1, 2, \cdots, n$. In this paper the author constructs examples of such systems for which $\Phi(t)$ is odd, or which violate the hypotheses of symmetry utilized previously, and which are parametrically unstable at $y=0$, $\lambda=0$. *L. Markus.*

Holodeckiĭ, R. A. Solution of linear differential equations of higher orders by means of infinite series. II. Solution of linear homogeneous differential equations of the third order with variable coefficients by means of infinite series. Leningrad. Inst. Inžen. Vod. Transp. Trudy **22** (1955), 203–227. (Russian)

Donskaya, L. I. On the structure of the solution of a system of n linear differential equations in the neighborhood of a regular singular point. Vestnik Leningrad. Univ. **9** (1954), no. 8, 55–64. (Russian)

Consider the system of n linear differential equations $\dot{X}=XP(t)$, where X is a fundamental matrix solution and the complex matrix $P(t)=\sum_{k=1}^{\infty}P^{(k)}t^{-k}$, where the $P^{(k)}$ are constant. Then on the complex t-sphere, $t=\infty$ is a singular point of the first kind. The author proves that there exists a constant matrix W, and a matrix $Z(t)$ analytic near $t=\infty$, such that $X=t^WZ(t)$. Similar results appear in the text of Coddington and Levinson [Theory of ordinary differential equations, McGraw-Hill, New York, 1955; MR **16**, 1022]. *L. Markus* (Princeton, N.J.).

Donskaya, L. I. On the structure of the solutions of a system of three linear homogeneous differential equations with an irregular singular point. Vestnik Leningrad. Univ. **1953**, no. 5, 15–64. (Russian)

This is a detailed study of the matrix equation $X=XP(t)$ with $P(t)=\sum_0^{\infty}P_nt^{-n}$, where P_n are constant 3×3 matrices. Using a device of Erugin [Trudy Mat. Inst. Steklov. **13** (1946); MR **9**, 509], the author obtains, according to the canonical form of P_0, non-singular solutions involving exponential functions, powers of t, and series that are uniformly convergent on $[T, \infty)$, $T\geqq0$, The case of 2-dimensional systems and of n-dimensional systems with P_0 restricted to have real and distinct eigenvalues have been treated analogously by Horošilov [Leningrad. Gos. Univ. Uč. Zap. **137**, Ser. Mat. Nauk **19** (1950), 180–197; Dokl. Akad. Nauk SSSR (N.S.) **72** (1950), 241–242; Prikl. Mat. Meh. **15** (1951), 37–54; MR **11**, 664; **12**, 610]. *H. A. Antosiewicz.*

Wasow, Wolfgang. The asymptotic theory of linear differential equations involving a parameter. Matematiche, Catania **10** (1955), 134–148.

This paper is a survey of the asymptotic theory of analytic differential systems of the form

$$y_i'=\sum_{k=1}^{n}a_{ik}(x, \varepsilon)y_k\quad(i=1, \cdots, n)$$

near a pole of the coefficients with respect to the parameter ε. The formal series expansions and Stokes' phenomenon are described. Airy's equation and Langer's method to show the validity of the expansions for the second order problem are discussed. For higher-order problems there is no general theory. Certain methods and results for special equations are sketched; in particular, the author's complete treatment of the Orr-Sommerfeld equation of hydrodynamic stability is very briefly outlined. *C. S. Morawetz* (New York, N.Y.).

★ **Asymptotic solutions of differential equations with turning points. Review of the literature.** Tech. Rep. 1. Department of Mathematics, California Institute of Technology, Pasadena, 1953. 16 pp.

Bibliography with brief reviews of the papers.

Flatto, L.; and Levinson, N. Periodic solutions of singularly perturbed systems. J. Rational Mech. Anal. **4** (1955), 943–950.

The paper discusses the vector systems

$$\frac{dx}{dt}=f(t, x, y, \varepsilon),\quad \varepsilon\frac{dy}{dt}=g(t, x, y, \varepsilon),$$

and establishes the existence of a unique periodic solution, of period T, for the system, which tends uniformly in t as $|\varepsilon|\to0$ to the postulated periodic solution of the reduced system

$$\frac{dx}{dt}=f(t, x, y, 0),\quad 0=g(t, x, y, 0),$$

also of period T. *G. E. Latta* (Stanford, Calif.).

Kurzweil, Jaroslav; and Vejvoda, Otto. On the periodic and almost periodic solutions of a system of ordinary differential equations. Czechoslovak Math. J. **5(80)** (1955), 362–370. (Russian. English summary)

Consider the real differential system

$$\dot{y}_s=f_s(y_1, y_2, \cdots, y_r, t)\quad(s=1, 2, \cdots, r),$$

where the continuous functions $f_s(y, t)$ have period ω in t for each fixed vector y. The authors consider the question of when the existence of a solution $y_s(t)$ with period Ω requires that Ω/ω is rational. The example $y''+y=(y^2+y'^2-1)\sin 2\pi t$ with solution $y=\sin t$ shows that

this is not always the case. Let N be the set of points $y^{(0)}$ for which $h_s(t)=f_s(y^{(0)}, t)$ are constant on $-\infty<t<\infty$. The authors then prove that if N is empty, Ω/ω is rational. In the course of the proof the analogous problem for differential systems with almost periodic coefficients is analyzed. *L. Markus* (Princeton, N.J.).

Morimoto, Hiroshi. On the perturbation of the linear system with constant coefficients possessing periodic solutions. Math. Japon. 3 (1955), 103–110.

The author considers the vector system $dz/dt=Az+\lambda g(z, t)$ where A is a constant square matrix. The vector g is twice differentiable with respect to the components of z and g is periodic in t of period 2π. Combining methods of Haag and of Coddington and Levinson, the author obtains results of the latter [Contributions to the theory of nonlinear oscillations, Princeton, 1952, vol II, pp. 19–35; MR **14**, 981]. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip; and Wintner, Aurel. On linear and nonlinear perturbations of linear systems of ordinary differential equations with constant coefficients. Trans. Amer. Math. Soc. **81** (1956), 1–24.

Part I contains two theorems on the asymptotic behaviour of the solutions of systems of ordinary differential equations with assertions which are stronger than those proved by Perron [Math. Z. **29** (1928), 129–160; **32** (1930), 703–728]. The first is a corollary of the other, but his assumptions are simpler: Let J be a constant d-by-d matrix and let $(\lambda-\lambda(1)^{h(1)}, \cdots, (\lambda-\lambda(g))^{h(g)}$, where $h(1)+\cdots+h(g)=d$, be the elementary divisors of J. Let $G(t)$ be a continuous d-by-d matrix on $0\leq t<\infty$ satisfying, as $t\to\infty$, $t^kG(t)\to 0$, where $K=\max(h(1), \cdots, h(g))$. Then the system (1) $y'=(J+G(t))y$, where y is a column vector, has d linearly independent solutions $y=y_{jl}(t)$, where $j=1, \cdots, g$ and $l=0, 1, \cdots, h(j)-1$, such that $y=y_{jl}(t)$ satisfies $\log|y(t)|=\mu t+(l+o(1))\log t$ with $\mu=\mu(j)$.

Part II is concerned with $O(1)$-variants of results of Part I for the case when J has only one elementary divisor. A specialisation of the results to the case when (1) is equivalent to a scalar differential equation of the dth order gives theorems related to results of A. Kneser on the equation $x''+f(t)x=0$.

Part III deals with analogues of the results of Part I and II for the case when the linear system (1) is replaced by a nonlinear system $y'=Jy+F(t, y)$.

In the Appendix, there are considered the conditions of Lonn [Jber. Deutsch. Math. Verein., **43** (1934), Abt. 1, 232–237] assuring that all solutions of $dy/dx=(-y+x+F^2(x, y))/(-x+F^1(x, y))$, in a sufficiently small circle $x^2+y^2\leq\varepsilon^2$, are tangent to the y-axis or that all solutions are spirals. Lonn's sufficient condition for spirals is considerably improved and both of Lonn's results are extended so as to apply to nonconservative systems $x'=-x+F^1(t, x, y)$, $y'=-y+x+F^2(t, x, y)$ as well. *M. Zlámal* (Brno).

★ **Rosenberg, R. M. On the stability of a nonlinear nonautonomous system.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 63–67. American Society of Mechanical Engineers, New York, 1955. $9.00.

The author investigates the stability of solutions of (*) $x''+a^2x+x^2f(t)=0$, where $a\neq 0$ is a real constant, for several types of functions $f(t)$. A solution of (*) is called stable if it can be written $x=\varepsilon[\sin(at+\phi)+\eta(t, \varepsilon)]$ and $\eta(t, \varepsilon)$ is bounded for all $t\geq 0$. Stability is deduced for sufficiently small ε if $f(t)$ is constant, and if $f(t)$ is positive (negative), bounded, and increasing (decreasing) for $t\geq 0$. If $f(t)$ is monotonic and exponentially unbounded, solutions are shown to be unstable. If $f(t)$ is periodic, stability is related to a Mathieu equation. *W. S. Loud.*

Barbašin, E. A.; and Skalkina, M. A. On stability in the first approximation. Prikl. Mat. Meh. **19** (1955), 623–624. (Russian)

The author considers the system

(1) $y_s'(t)=Y_s(t, y_1, \cdots, y_n)+R_s(t, y_1, \cdots, y_n)$ $(s=1, \cdots, n)$

to be a perturbation of the system

(2) $x_s'(t)=Y_s(t, x_1, \cdots, x_n)$,

where (a) Y_s and R_s satisfy Lipschitz's condition in the region $|y_s|\leq H$, $t\geq 0$, and (b) the zero function is a solution of (1) and (2). It then follows that, if $|y_s|\leq y\leq H$ and $t\geq 0$,

$$|R_s(t, y_1, \cdots, y_n)|\leq Ry$$

for some R. The paper consists of the statement and proof of the following result: if each solution of (2) satisfies

$$|x_s(t)|\leq Bxe^{-\alpha(t-t_0)} \text{ for all } t\geq 0,$$

where x is any number with the property that $|x_s(t_0)|\leq x\leq H$, $t_0\geq 0$, and B and α are positive constants that do not depend on $t_0, x_1(t_0), \cdots, x_n(t_0)$, then for R sufficiently small the zero solution of (1) is assymptotically stable, and this stability is uniform relative to $t_0, y_1(t_0), \cdots, y_n(t_0)$. The proof establishes the theorem for the case $4Rn^2Te^{n(2L+R)T}<1$, where $T=\alpha^{-1}\ln 4B$ and L is the Lipschitz constant of $Y=(Y_1, \cdots, Y_n)$. *J. P. LaSalle.*

Turán, P. On the instability of systems of differential equations. Acta Math. Acad. Sci. Hungar. **6** (1955), 257–270. (Russian summary)

Consider the complex differential system

1) $\dot{X}_\nu(t)=\sum_{j=1}^{n} a_{\nu j}X_j(t)+w_\nu(x_1, \cdots, x_n, t)$ $(\nu=1, 2, \cdots, n)$

where the eigenvalues λ_ν of the matrix $(a_{\nu\mu})$ have real parts with $\min_{\nu=1,2,\cdots,n}\mathrm{Re}(\lambda_\nu)=L>0$. If $w_\nu(x, t)$ consists of "higher-order" terms near $X_\nu=0$, then the differential system is completely unstable. This means that there is a $c>0$ such that for each solution of (1) with

$$0<\sum_{\nu=1}^{n}|X_\nu(0)|^2<c$$

one has $\sum_{\nu=1}^{n}|X_\nu(t_1)|^2\geq c$ for some $t_1>0$ (provided the solution is defined for a sufficiently long time duration).

The author weakens the usual assumptions on w_ν as follows. Let

$$\max_{\nu,j}|a_{\nu j}|=C \text{ and } D=\frac{n\log(4C)+\log(16n)}{2L}.$$

Theorem. For some $a>0$, let the functions $w_\nu(y, t)$, be defined and continuous in the domain $t\geq 0$, $|y_1|\leq a, \cdots, |y_n|\leq a$. If

$$\frac{\sum_{k=1}^{n}|w_k(y, t)|^2}{\sum_{k=1}^{n}|y_k|^2}\leq\frac{1}{30nD^2}e^{-4nCD},$$

then the solution $X_\nu(t)=0$ is completely unstable. The proof of this theorem, and another rather more technical result, are based on the author's method of estimating trigonometric sums. *L. Markus* (Princeton, N.J.).

Otrokov, N. F. Generation of limit cycles in algebraic differential systems. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 26–28. (Russian)

The author studies the stability of the number of limit cycles of the system $dx/dt=P(x, y)$, $dy/dt=Q(x, y)$, where P and Q are polynomials, under perturbations in the coefficients of P and Q. Under various assumptions concerning P and Q, upper and lower bounds for the number of limit cycles of the perturbed equations are given.

R. Bellman (Santa Monica, Calif.).

★ **Sethna, Patarasp Rustomji. Steady state motion of one and two degrees of freedom vibrating systems with a nonlinear restoring force.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 69–78. American Society of Mechanical Engineers, New York, 1955. $9.00.

Author's Summary: In this paper are studied the steady state motions of one-and-two-degrees-of-freedom vibrating systems with quadratic or cubic nonlinear restoring forces and under the influence of external sinusoidal excitations. The first and second approximations are given for some one-degree-of-freedom undamped systems, and first approximations are given for the undamped and damped cases of a two-degrees-of-freedom system. The adequacy of the methods employed is demonstrated by comparing the results obtained by them with those obtained from an electronic analog computer. *W. S. Loud* (Cambridge, Mass.).

Giovannini, Gigliana. Sulla sincronizzazione dei sistemi non lineari a più di due gradi di libertà. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 371–376.

The author raises the question of whether or not the stroboscopic method is applicable to problems concerning nonlinear systems having more than two degrees of freedom. She considers a particular system having three degrees of freedom; and finds that, in order for the method to be applicable, a certain complicated system of equations must have real solutions. In one special case the existence of suitable solutions is evident; and on the basis of this the author tentatively answers her question in the affirmative. *L. A. MacColl* (New York, N.Y.).

Litvin-Sedoĭ, M. Z. On stability of periodic motions of a system of automatic regulation. Moskov. Gos. Univ. Uč. Zap. **172** (1954). Meh. **5**, 163–176. (Russian)

The system of automatic regulation is described by the system of differential equations

$$\begin{aligned} &(Y^2D^2+XD+l)\varphi-\psi=0,\\ &(SD+1)\theta-SD\varphi=0,\\ &D^2W(T^2D+U)\varphi+kWD\theta+h\psi=\mu\Phi(\psi), \end{aligned} \tag{1}$$

where $D=d/dt$, and Y, X, l, S, W, T, U, k, μ, and h are constants; the function $\mu\Phi$ measures the nonlinearity of the servo-motor.

By the use of "small-parameter" methods, expressions are obtained which for sufficiently small μ provide estimates of the characteristic exponents of periodic solutions. Sufficient conditions for the stability of periodic solutions are obtained from these estimates using the fact that a periodic solution of (1) is asymptotically [orbitally] stable if all but one of its characteristic exponents have negative real parts. These sufficient conditions contain the somewhat indefinite restriction that the parameter μ be sufficiently small.

J. P. LaSalle (Notre Dame, Ind.).

Popov, E. P. On the use of the method of harmonic linearization in the theory of automatic control. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 211–214. (Russian)

The object of this paper is threefold: a) to establish the existence of the small parameter for a class of nonlinear problems that can be solved by the method of harmonic linearization, b) to show the relation between this parameter and the intuitive physical concepts that arise in the application of the method, and c) to give generalizations of the method.

The nonlinear differential equation considered is of the form

$$Q(p)x+R(p)F(x, px)=0 \quad (p=d/dt),$$

where $Q(p)$ and $R(p)$ are polynomials, while $F(x, px)$ is a given nonlinear function. The author establishes a certain inequality involving the coefficients of the given equation. On the basis of this inequality he establishes the existence of the small parameter, which is used as a criterion for the validity of the application of the method of harmonic linearization to the given problem. A physical interpretation of the small parameter is given in terms of the amplitude and period of the solution of the approximating system. Finally the author extends the class of problems to which the method of harmonic linearization is applicable. He points out that arguments analogous to those given here can be used to establish the validity of approximate methods for the analysis of transient oscillatory processes in nonlinear control systems. *H. P. Thielman.*

★ **Petrov, B. N.; Popov, E. P.; Voronov, A. A.; and Hramoĭ, A. V. Development of the theory of automatic control in the USSR.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 13–50. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

★ **Letov, A. M. The status of the problem of stability in the theory of automatic control (a survey).** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 79–104. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)
Expository paper.

★ **Aĭzerman, M. A. The problem of determining periodic regimes in systems of automatic control (a survey).** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 105–130. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)
Expository paper.

★ **Lur'e, A. I. Lyapunov's direct method and its application to the theory of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 142–148. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)
Expository paper with a substantial bibliography.

★ **Stebakov, S. A. Application of the concept of phase space for investigation of systems of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 149–165. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

The author discusses in a general way a number of automatic control systems and their geometric representation in the Euclidean space of dependent variables.

H. A. Antosiewicz (Washington, D.C.).

★ **Gantmaher, F. R. On structural stability of a single-contour system of automatic control with action on the derivatives.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 315–323. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

The control system studied here is a single loop system characterized by the relations

$$(1)\quad d_\nu(p)x_\nu=k_\nu(p)x_{\nu-1}\ (\nu=1, 2, \cdots, m;\ x_0=-x_m),$$

where $d_\nu(p)$ and $k_\nu(p)$ are time-derivative operators of the form ap^2+bp+c. Stability of the system is then studied from a consideration of its characteristic equation, which is left in the form:

$$(2)\qquad D(p)+K(p)=0,$$

$$(3)\quad D(p)=d_1(p)\cdots d_m(p),\ K(p)=k_1(p)\cdots k_m(p).$$

Depending upon the additional assumptions made for the polynomials $D(p)$ and $K(p)$, necessary and sufficient conditions can be stated under which (2) will be a Hurwitz polynomial. There are six such stability theorems given (no proofs), of which the following is typical. (*) If the system (1) is of "normal type" and $K(p)$ is a Hurwitz polynomial, then necessary and sufficient conditions for stability are realized when $n_2\geqq\sigma+\tau_1-1$ and one of the following inequalities is satisfied:

$n_2=0$	$n_2>0$ and odd	n_2 even
μ_1 odd, $n>4\varrho_1$	$n>4\varrho_1-1$	$n>4\varrho_1-2$
μ_1 even, $n>4\varrho_1$	$n>4\varrho_1$	$n>4\varrho_1+1$.

In (*), $n=n_1+n_2$, n_1=degree of $D(p)$, n_2=degree of $K(p)$; σ is the number of zeros of $D(p)$ equal to zero, τ_1 the number of d_ν factors with $c<0$, λ_1 the number of zeros of $D(p)$ having negative real parts, $\mu_1=n_1-\lambda_1$, $\varrho_1=[\frac{1}{2}\mu_1]$. In the factors d_ν and k_ν, at least one of a, b, c is non-zero, the first non-zero one in d_ν being assumed positive; also for k_ν, $c>0$. "Normal type" implies the non-existence of operators d_ν with $b\leqq 0$, $c<0$ and of operators k_ν with $a<0$, $b\leqq 0$.

A final theorem (Th. 7) giving necessary (but not sufficient) conditions for the stability of (1) without additional assumptions on $D(p)$, and $K(p)$ is also given but is far too detailed to reproduce here.

The paper is concluded with an example showing how, through use of the appropriate theorem, an unstable system for which $K(p)\equiv$const can be rendered stable by introducing "derivative control", the new $K(p)$ being of the form $bp+c$, $(b>0)$. Some of the theorems stated should ultimately find their way into American books on control-system synthesis.

J. F. Heyda.

Barrett, J. F.; and Coales, J. F. An introduction to the analysis of non-linear control systems with random inputs. Proc. Inst. Elec. Engrs. C. **103** (1956), 190–199.

Latyševa, K. Ya. Finding finite solutions for linear differential equations of a special form. Kiïv. Derž. Univ. Nauk Zap. **12** (1953), Mat. Sb. no. **7**, 95–100. (Russian)

The author finds conditions on the real parameters a_0, a_1, b_0, b_1, c_0, c_1 in order that the linear differential equation

$$x^2(a_0+a_1x)y''+x(b_0+b_1x)y'+(c_0+c_1x)y=0$$

has a solution of the form

$$y=x^\alpha(A_0+A_1x+\cdots+A_qx^q).$$

Here $q\geqq 0$ is an integer and α, A_0, $\cdots$, A_q are real numbers.

L. Markus (Princeton, N.J.).

Bognár, J.; and Freud, G. Berechnung der gequantelten Energiewerte eines im elektrischen Felde rotierenden Dipols. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 239–251 (1955). (Hungarian. Russian and German summaries)

The second author has proved [Acta Phys. Acad. Sci. Hungar. **1** (1952), 325–328; MR **15**, 428] the following theorem: Let $y=\sum_0^\infty r_nx^n$ be a solution of the differential equation $(x^2-1)y''+2xy'+P(x)y=0$, where $P(x)$ is a polynomial. Then $\varrho_0=\lim(2nr_{2n})$, $\varrho_1=\lim((2n+1)r_{2n+1})$, $n\to\infty$, exist, and y is bounded for $-1\leqq x\leqq 1$ if and only if $\varrho_0=\varrho_1=1$. He also estimated $2nr_{2n}-\varrho_0$ and

$$(2n+1)r_{2n+1}-\varrho_1.$$

In the present paper, the authors apply Freud's result to the differential equation $(1-x^2)f''(x)-2xf'(x)+(\alpha+\beta x)f(x)=0$ which arises in the physical problem indicated in the title. Here β is a given parameter, and characteristic values of α (energy levels) are to be determined for which the differential equation has a bounded solution. This leads to a condition $\gamma_0\delta_1-\gamma_1\delta_0=0$, where γ_0, γ_1 and δ_0, δ_1 are limits resembling ϱ_0, ϱ_1 formed for two solutions of the differential equation. The condition is an equation between α and β, and this equation the authors solve numerically, giving tables of approximation of the characteristic values and energy levels.

A. Erdélyi (Pasadena, Calif.).

Marčenko, V. A. Theorems of Tauberian type in spectral analysis of differential operators. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 381–422. (Russian)

This work is devoted to improving and generalizing the methods previously used by the author [Trudy Moskov. Mat. Obšč. **1** (1952), 327–420, pp. 363–375; MR **15**, 315] for obtaining asymptotic formulas for the spectral function of one-dimensional differential operators of the second order. The statements of the results are too complicated to give here, but the basic problem of Tauberian type which is considered may be formulated as follows: Let, for some integer $n\geqq 0$, there be given functions ϱ and σ on $-\infty<\lambda<\infty$ which are non-decreasing and continuous from the left, and which satisfy the conditions: (I) for some function G which is summable on an interval $(-h, h)$, and for all functions $f\in C^\infty$ which vanish outside $(-h, h)$,

$$\int_{-\infty}^{\infty}E_f(\lambda)d\varrho(\lambda)=\int_{-\infty}^{\infty}E_f(\lambda)d\sigma(\lambda)+\int_{-\infty}^{\infty}f^{(n)}(x)G(x)dx,$$

where $E_f(\lambda)=\int_{-\infty}^{\infty}f(x)e^{-i\lambda x}dx$; (II) σ satisfies

$$\limsup_{|a|\to\infty}\frac{\sigma(a+\lambda_0)-\sigma(a)}{|a|^n}=\sigma^*(\lambda_0)<\infty$$

for a certain $\lambda_0>0$. It is required to estimate $|\varrho(\lambda)-\sigma(\lambda)|$ for large $|\lambda|$ in terms of the differentiability properties of G, or to make some other conclusion concerning the proximity of ϱ to σ as $|\lambda|\to\infty$, such as to estimate $\int_a^{a+1}[\varrho(\lambda)-\sigma(\lambda)]d\lambda$, as $|a|\to\infty$. The results obtained are more general and more precise than those previously given by the author, and by B. M. Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 269–284; MR **15**, 316]. In the last paragraph some applications are made to ordinary differential operators of the second order. After the paper had gone to the printer a paper of B. M. Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 33–58; MR **16**, 1027] appeared in which certain cases of these latter results were proved. The author plans to give more complete applications to ordinary and partial differential operators in a separate paper. *E. A. Coddington.*

Lomonosov, M. I. On expansion in eigenfunctions of the operator

$$-\frac{d}{dy}\left[p(y)\frac{d}{dy}U\right]+q(y)U.$$

Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 412–415. (Russian)

The author considers on $0\leqq y<c\leqq\infty$ the differential operator L defined by $L(U)=-(pU')'+qU$, where p is continuous and positive, q is real and summable on each subinterval $[0, d]$, $d<c$. The self-adjoint operator in $L^2[0, c)$ determined by this L and the boundary condition $p(0)U'(0)=\theta U(0)$ (and, in the limit-circle case at c, a similar condition at c) is denoted by L_θ. Corresponding to this L_θ is a non-decreasing spectral function σ which determines an isometry between $L^2[0, c)$ and $L^2(\sigma)$. Assuming that p has only one continuous derivative, the author gives an asymptotic formula for σ, and states an equiconvergence theorem with the ordinary Fourier integral expansion. A preliminary substitution $x=\int_0^y p^{-\frac{1}{2}}(t)dt$ is used, and then it is stated that the proofs of the main results follow from theorems of Tauberian type proved by V. A. Marčenko [see the preceding review]. *E. A. Coddington* (Los Angeles, Calif.).

Sargsyan, I. S. Summability of the derivatives of an expansion in eigenfunctions of a Sturm-Liouville operator. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 821–824. (Russian)

The equation $y''+(\lambda-q)y=0$ is considered on the half-line $(0, \infty)$, where q is real and summable on each finite subinterval. The differential equation, together with an initial condition at the origin, is assumed to constitute a problem with non-negative spectrum. The author extends some equiconvergence results of B. M. Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 33–58; MR **16**, 1027]. The final result states that if q has a bounded $(k-1)$-st derivative in each finite subinterval of $(0, \infty)$ then the difference between the Riesz mean of order k for the kth derivative of the eigenfunction expansion of a function $f\in L_2(0, \infty)$, and for the kth derivative of the corresponding Fourier cosine integral expansion, tends to zero uniformly on each finite interval. *E. A. Coddington.*

Titchmarsh, E. C. Some theorems on perturbation theory. V. J. Analyse Math. **4** (1954/55), 187–208.

Let $q(x)=\frac{1}{4}(n^2-1)x^{-2}+cx^{-1}$ where n is a positive integer and c a positive constant and let $\sigma(x)=-x$. The equation $\Psi''+(\lambda-q(x))\Psi=0$ with $0<x<\infty$ has a discrete spectrum. If q is replaced by $Q=q+\varepsilon\sigma$, there is a continuous spectrum. Let $\varrho(\lambda, \varepsilon)$ be the spectral weighting function so that the Parseval theorem for a real function $f(x)$ of $L^2(0, \infty)$ and transform $g(\lambda, \varepsilon)$ takes the form

$$\int_0^\infty f^2(x)dx=\int_{-\infty}^\infty g^2(\lambda, \varepsilon)d\sigma(\lambda, \varepsilon).$$

The author shows that if E is a variable set of intervals which converge to the points of the discrete spectrum as $\varepsilon\to 0$ then

$$\int_0^\infty f^2(x)dx=\int_E g^2(\lambda, \varepsilon)d\varrho(\lambda, \varepsilon)+o(1)$$

as $\varepsilon\to 0$. The interval of E containing λ_n, the nth eigenvalue for the case $\varepsilon=0$, can be taken as $(\lambda_n+\varepsilon\lambda_n^{(1)}-\varepsilon^p, \lambda_n+\varepsilon\lambda_n^{(1)}+\varepsilon^p)$ where $p>1$

$$\lambda_n^{(1)}=\int_0^\infty\sigma(x)\Psi_n^2(x)dx$$

where Ψ_n is the nth eigenfunction for $\varepsilon=0$.
N. Levinson (Cambridge, Mass.).

See also: Gorelik, p. 813; Roĭtenberg, p. 813; Piven', p. 814; Dobronravov, p. 820; Terracini, p. 820; Smir'now, p. 833; Wintner, p. 846; Antosiewicz, p. 874; Kislov, p. 902.

Partial Differential Equations

Hörmander, Lars. On the theory of general partial differential operators. Acta Math. **94** (1955), 161–248.

Ce travail contient de nombreux résultats importants et entièrement nouveaux.

Notations: sur Ω ouvert de R^ν $(x=(x^1, \cdots, x^\nu)\in\Omega)$ on donne un opérateur différentiel

$$\mathcal{P}=\sum a^\alpha(x)D_\alpha,\quad a^\alpha=a^{\alpha_1\cdots\alpha_\nu},\quad D_\alpha=D_{\alpha_1}\cdots D_{\alpha_\nu},$$

$D_{\alpha_k}=\partial/i\partial x^{\alpha_k}$. Soit $P(x, \xi)$ le polynome en $\xi=(\xi_1, \cdots, \xi_\nu)$ correspondant; si $P^\alpha(x, \xi)$, x fixé, est une dérivée en ξ du polynome $P(x, \xi)$, on désigne par $\mathcal{P}$ l'opérateur différentiel correspondant.

On désigne par $\mathcal{D}_P$ l'espace des (classes de) fonctions $u\in L^2(\Omega)$ telles que $\mathcal{P}u$ (pris au sens des distributions sur Ω) soit dans $L^2(\Omega)$, espace muni de sa structure hilbertienne naturelle. Soit $\mathcal{D}_{P_0}$ l'adhérence dans $\mathcal{D}_P$ de l'espace C_0^∞ des fonctions indéfiniment différentiables à support compact dans Ω. L'A. désigne par P (resp. P_0) l'opérateur $\mathcal{P}$ défini sur $\mathcal{D}_P$ (resp. $\mathcal{D}_{P_0}$); P (resp. P_0) est dit maximal (resp. minimal). Soit $\mathcal{P}$ et $\mathcal{Q}$ deux opérateurs différentiels. On dit que $\mathcal{P}$ est plus fort que $\mathcal{Q}$ sur Ω si $\mathcal{D}_{P_0}\subset\mathcal{D}_{Q_0}$ (espaces construits sur Ω).

Théorème 1. Une condition nécessaire et suffisante pour que $\mathcal{P}$ soit plus fort que $\mathcal{Q}$ sur Ω borné (cas des coefficients constants) est qu'il existe une constante C telle que

$$\tilde{Q}(\xi)\leqq C\tilde{P}(\xi),$$

où $\tilde{P}(\xi)^2=\sum|P^\alpha(\xi)|^2$ (id. pour $\tilde{Q}(\xi)$). Cette condition est indépendante de Ω. La démonstration repose essentiellement sur l'important Théorème 2. Soit Ω un ouvert borné, $\mathcal{P}$ à coefficients constants (l'A. écrit alors $\mathcal{P}=P(D)$. Quel que soit α, il existe une constante k telle que, pour tout $u\in C_0^\infty(\Omega)$,

$$\|\mathcal{P}u\|\geqq k\|P^\alpha u\| \tag{1}$$

($\|f\|$ est la norme dans $L^2(\Omega)$).

La démonstration du théorème 2 repose sur des intégrales d'énergie (méthode de Friedrichs, Lewy et Leray dans les cas hyperboliques). Dans le cas particulier $\alpha=0$, l'A. donne une autre démonstration, utilisant la transformation de Laplace et un lemme voisin d'un lemme de Malgrange [C.R. Acad. Sci. Paris **237** (1953), 1620–1622; MR **15**, 626]. La méthode des intégrales d'énergie permet d'attaquer le cas des opérateurs à coefficients variables. Soit $p(x, \xi)$ la partie principale du polynome $P(x, \xi)$; on dit que $\mathcal{P}$ (ou p) est de type principal si les $\partial p(x, \xi)/\partial \xi_i$ ne peuvent s'annuler simultanément pour $x \in \Omega$, $\xi \neq 0$.

Théorème 3. Soit $\mathcal{P}$ à coefficients variables continus, de type principal, d'ordre m, $p(x, \xi)$ réel à coefficients une fois continûment différentiables. Alors tout point de Ω a un voisinage ω dans lequel

$$\|\mathcal{P}u\|^2 \geqq k \sum_{|\alpha|<m} \|D_\alpha u\|^2, \quad u \in C_0^\infty(\omega). \tag{2}$$

Des théorèmes 2 et 3 résulte que $\mathcal{P}$ applique $\mathcal{D}_P$ sur L^2 d'où facilement l'existence de problèmes aux limites bien posés. Le théorème 2 entraîne aussi l'existence d'une solution élémentaire (théorème de Malgrange, loc. cit.). Le théorème 3 entraîne l'existence locale d'un noyau élémentaire. Par utilisation des inégalités (1), l'A. résout ensuite (entre autres) les problèmes suivants: condition pour que Qu soit dans $L^p(\Omega)$ (resp. continue après modification éventuelle sur un ensemble de mesure nulle, etc.) pour tout $u \in \mathcal{D}_{P_0}$; condition nécessaire et suffisante pour que l'injection de $\mathcal{D}_{P_0}$ dans $\mathcal{D}_{Q_0}$ soit complètement continue.

Un opérateur $\mathcal{P}$ (toujours à coefficients constants) est dit complet (Gårding) si "η réel et $P(\xi+t\eta)=P(\xi)$ pour tout ξ et t" entraînent $\eta=0$. Alors si u est dans le domaine minimal de $\mathcal{P}^n$ quel que soit n, $\mathcal{P}$ étant complet, u est indéfiniment différentiable dans Ω. Un problème naturel est maintenant de comparer $\mathcal{D}_P$ et $\mathcal{D}_Q$. On a: Théorème 4. Une condition nécessaire et suffisante pour que $\mathcal{D}_P \subset \mathcal{D}_Q$ est que ou bien $\mathcal{Q}=a\mathcal{P}+b$, a, b=constantes, ou bien $\mathcal{P}$ et $\mathcal{Q}$ sont deux opérateurs différentiels ordinaires, avec ordre $\mathcal{Q} \leqq$ ordre $\mathcal{P}$. La démonstration de ce théorème fait appel à des résultats non triviaux de géométrie algébrique et utilise entre autres une idée de B. L. van der Waerden.

Il résulte du Théorème 2 que: $u \in \mathcal{D}_{P_0}$ et $\varphi \in C_0^\infty$ entraîne $\varphi u \in \mathcal{D}_{P_0}$. On dit que $\mathcal{P}$ est de type local si la même propriété a lieu pour $u \in \mathcal{D}_P$. L'A. donne cinq conditions équivalentes (et non trivialement équivalentes) nécessaires et suffisantes pour que $\mathcal{P}$ soit de type local.

La terminologie suivante a été adopteé au Colloque International sur les équations aux dérivées partielles, Nancy, Avril 1956: un opérateur différentiel D est dit hypo-elliptique si $Du=S$ entraîne que u est indéfiniment différentiable en même temps que S. Alors: Théorème 5. Une condition nécessaire et suffisante pour que $\mathcal{P}$ soit hypo-elliptique est que $\mathcal{P}$ soit complet et de type local. La démonstration repose entre autres sur la construction d'une solution élémentaire d'un opérateur complet et vérifiant l'une des cinq conditions équivalentes dont on parle çi dessus. A l'aide des propriétés de la solution élémentaire ainsi construite, l'A. donne des estimations pour les majorations des dérivées successives des solutions de $Pu=0$. Il généralise ainsi considérablement les résultats de Holmgren pour l'équation de la chaleur.

Signalons encore que l'A. démontre l'existence d'une "solution nulle" dans toute direction caractéristique, $\mathcal{P}$ étant quelconque à coefficients constants, et qu'il étudie brièvement la théorie spectrale des opérateurs self-adjoints hypo-elliptiques. *J. L. Lions* (Nancy).

Ehrenpreis, Leon. General theory of elliptic equations. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 39–41.

D=distribution à support compact sur R^n; soit $P(z)=P(x+iy)$ la transformée de Fourier de D, V la variété des zéros de P dans C^n; pour $x \in R^n$, $\varrho(x)$=distance de x à V. Condition a: $\lim_{|x|\to\infty} (\varrho(x))^{-1} \log |x|=0$. Condition c: $\varrho(x) \geqq A|x|$ pour $|x|$ assez grand. L'A. annonce sans démonstration le théorème: Une condition nécessaire et suffisante pour que toute distribution S, solution de $D*S=0$, soit indéfiniment différentiable (resp. analytique) est que la condition a (resp. c) ait lieu. Dans le cas c, D est à support un point. *J. L. Lions* (Nancy).

Diaz, J. B.; and Ludford, G. S. S. Reflection principles for linear elliptic second order partial differential equations with constant coefficients. Ann. Mat. Pura Appl. (4) **39** (1955), 87–95.

Let D be a domain in R^n with a hyperplane P, say, $x_1=0$, as the plane of symmetry. Denote by D^+, D^- and d respectively the parts of D with $x_1>0$, $x_1<0$ and $x_1=0$. Extending the reflection principle of harmonic functions to the Helmholtz equation: $H_n(u)=\partial^2 u/\partial x_1^2+\cdots+\partial^2 u/\partial x_n^2+\lambda u=0$, λ=const, the authors prove the following two theorems. Let u be a twice differentiable solution of $H_n(u)=0$ in D^+ with the boundary conditions: 1) $\lim_{x\to\bar{x}} u=0$ and 2) $\lim_{x\to\bar{x}} \partial u/\partial n+ku=0$, where k=const, n is the direction normal to P, x lies in D^+ and $\bar{x}$ in d. The solution with 1) can be extended analytically beyond d into the whole D^- by a simple reflection on P. In the case of 2) one finds first by a simple reflection the extension of the function v, where $v=\partial u/\partial x_1+ku$, beyond d into the whole D^-. The extension of u is then obtained by an integration along the x_1-direction, so that the new domain of u consists of d and those points in D^- which can be connected to d by line segments parallel to x_1 axis and lying in D^-. That in general the new domain can not be the whole D^- is demonstrated by an example. Similar results hold for problems with $\lim_{x\to\bar{x}} \partial u/\partial s+ku=0$, where s is any direction not tangential to P. The general case of elliptic equations of second order with constant coefficients is reduced to the Helmholtz equation by coordinate transformations. *Y. W. Chen.*

Arešev, M. S. On continuous dependence of the solutions of a system of linear partial differential equations on the initial data and on connections of the Dirichlet problem and the Neumann problem with the Cauchy problem. Akad. Nauk Uzbek. SSR Trudy Inst. Mat. Meh. **13** (1954), 129–134. (Russian)

The author notes that the solution of the Cauchy problem for a linear system of equations with analytic coefficients and initial data prescribed on a non-characteristic surface depends continuously on the initial data in the complex domain but (as is well-known) not necessarily in the real domain. He observes that in some cases it is possible to impose relations on the initial data in order that the solution of the Cauchy problem solve simultaneously a suitable Dirichlet or Neumann problem. *R. Finn* (Los Angeles, Calif.).

Aronszajn, Nachman. Sur l'unicité du prolongement des solutions des équations aux dérivées partielles elliptiques du second ordre. C. R. Acad. Sci. Paris **242** (1956), 723–725.

Let A be an elliptic differential operator of the second order defined in a region D of real n-dimensional space. Let its coefficients be bounded and let its principal part

$\sum a^{ij}\partial_i\partial_j$ ($\partial_i=\partial/\partial x^i$) be positive definite with coefficients whose second derivatives satisfy a Lipschitz condition. Consider the inequality

$$(1) \qquad |Au(x)|\leq M(|u(x)|+\sum|\partial_k u(x)|),$$

(M a constant, $x\in D$). The following important theorem is announced. If $u\in C^2$ satisfies (1) and has a zero of infinite order in D, then u vanishes identically. The first theorem of essentially this kind was proved by Carleman [same C. R. **197** (1933), 471–474]. The special case when A has constant coefficients has been considered by C. Müller [Comm. Pure Appl. Math. **7** (1954), 505–515; MR **16**, 42] and E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626]. The proof is based on the inequality

$$(2) \quad \mathrm{ch}^2\int_S r^{-2\alpha}|Bu|^2dv\geq\int_S r^{-2\alpha}(\sum b^{ij}\partial_i u\partial_j u+|u|^2)\,dv$$

used in a special case by Heinz. Here u is a solution of (1) having a zero of infinite order at a point x_0 and vanishing outside $S\in x_0$. The operator B is Beltrami's operator in a metric b given by $b^{ij}=e^{2\nu\varrho}a^{ij}$, where ϱ is the geodesic distance between x_0 and x in the metric $a=(a^{ij})$ and ν is a positive number chosen so large that b has positive curvature in a neighborhood of x_0. The function r is the geodesic distance between x_0 and x in the metric b and S is the geodesic sphere $r<h$. The number α is arbitrary positive and c is a constant depending only on a and x_0. Finally, dv is the invariant volume element given by b. More general that (2) is the inequality

$$\mathrm{ch}^{2-2\omega}(1-\omega)^2\int_S r^{-2\alpha}|Bu|^2dv\geq$$
$$\int_S r^{-2\alpha}(r^{-2\omega}b^{ij}\partial_i u\partial_j u+r^{-2\omega-2}|u|^2)dv$$

which holds when $0\leq\omega<1$ and makes it possible to extend the theorem to functions u satisfying the inequality

$$|Au(x)|\leq M(|x-x_0|^{-\omega}\sum|\partial_k u(x)|+|x-x_0|^{-\omega-1}|u(x)|).$$

A corollary of the theorem is the uniqueness of Cauchy's problem for quasi-linear elliptic equations under relatively mild regularity assumptions. A corresponding uniqueness theorem is also announced for the generalized Monge-Ampère equation $\det(\partial_i\partial_j u)=F(x, u, \partial_i u, \cdots, \partial_n u)$. A solution is called elliptic if it is strictly concave or convex in D. Two elliptic solutions of class C^4 with fourth derivatives satisfying a Lipschitz condition and having the same Cauchy data on a sufficiently regular portion of the boundary of D coincide in D, provided F satisfies a Lipschitz condition. *L. Gårding* (Lund).

Hartman, Philip; and Wintner, Aurel. On the local behavior of solutions of non-parabolic partial differential equations. III. Approximations by spherical harmonics. Amer. J. Math. **77** (1955), 453–474.

[For parts I–II see same J. **75** (1953), 449–476; **76** (1954), 351–361; MR **15**, 318; **16**, 256.] This paper is concerned with the asymptotic behavior at zeros or at isolated singularities of the solution of a linear or non-linear elliptic partial differential equation in $n\geq 3$ independent variables of the form $\Delta u+g(x, u, \nabla u)=0$, where g is a continuous function for $|x|\leq 1$, $|u|\leq$const, $|\nabla u|\leq$const subject to an inequality of the form $|g(x, u, \nabla u)|\leq\text{const}\times(|u|+|\nabla u|)$. The methods are generalizations of those of C. Müller [Comm. Pure Appl. Math. **7** (1954), 505–516; MR **16**, 42]. The results apply to "C^1-solutions" of the equation, functions of class C^1 that can be represented as the sum of an harmonic function and the Newtonian potential of $-g$. The first result states that, if $u(0)=0$, then either $u\equiv 0$ for $|x|\leq 1$ or else for some spherical harmonic $S_N(e)$ of order N on the unit sphere $r=|x|=1$, the relations $u(x)=r^N S_N(x/r)+o(r^N)$ and $\nabla u(x)=\nabla(r^N S_N(x/r))+o(r^{N-1})$ hold uniformly in x/r as $r\to 0$. (Analogous relations are given for the higher derivatives of u, provided g is sufficiently smooth.) As a corollary, a theorem of unique continuation is proved asserting that, if $u(x)=o(r^n)$ for $n=1, 2, \cdots$ as $r\to 0$, then $u(x)\equiv 0$ for $|x|\leq 1$. This unique continuation theorem, an extension of Müller's, has been given independently by E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626] and, quite recently, has been generalized by N. Aronszajn [in the paper reviewed above] to arbitrary elliptic equations of second order. The other main results pertain to C^1 solutions of the equations which are defined on the punctured disk $0<|x|\leq 1$ and are such that, as $r\to 0$, (a) $u(x)=o(1/r)$, (b) $u(x)\to\infty$, or (c) $u(x)=o(r^{-N-2})$ and $\nabla u(x)=o(r^{-N-3})$ for some integer $N\geq 0$. The solution in case (a) is shown to have a removable singularity at $x=0$, in case (b) to be of the form $c/r+O(|\log r|)$ with $\nabla u(x)=\nabla(c/r)+O(1/r)$ and in case (c) to be such that $u(x)=r^{-N-1}S_N(x/r)+o(r^{-N-1})$ and $\nabla u=\nabla(r^{-N-1}S_N(x/r))+o(r^{-N-2})$ for some N uniformly in x/r. *A. Douglis.*

Nikol'skiĭ, S. M. Properties of certain classes of functions of several variables on differentiable manifolds and their application to variational problems. Acta Math. Acad. Sci. Hungar. **5** (1954), supplementum, 61–70. (Russian).

Translation of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3 (1953), 243–252; MR **16**, 453. *F. V. Atkinson.*

Ladyženskaya, O. A. A simple proof of the solvability of the fundamental boundary problems and of a problem on eigenvalues for linear elliptic equations. Vestnik Leningrad. Univ. **10** (1955), no. 11, 23–29. (Russian)

Sur un ouvert Ω de R^n, W^2 est l'espace des fonctions u telles que $D^p u\in L^2(\Omega)$ pour tout p avec $|p|\leq 2$; on pose

$$\|u\|^2_{W^2}=\sum\|D^p u\|^2_{L^2},\quad |p|\leq 2.$$

Soit dans Ω l'opérateur

$$L=\sum_{i,j}a_{ij}(x)D^2_{x_i x_j}+\sum_i a_i(x)D_{x_i}-a(x)$$

elliptique: $\sum a_{ij}(x)\xi_i\xi_j\geq\alpha\sum_i\xi_i^2$, $\alpha>0$; a_{ij} continue dans $\overline{\Omega}$, de dérivées (distributions) en x_i dans L^∞, a_i et a dans L^∞. Démonstration rapide du théorème suivant: Ω est une boule; si $a(x)\geq$ constante convenable, alors il existe une constante c telle que $\|Lu\|_{L^2}\geq c\|u\|_{W^2}$, pour tout u dans W^2, nulle au bord. [Cf. aussi l'A. Le problème mixte pour une équation de type hyperbolique, Gotsehizdat, Moscow, 1953; MR **17**, 160; pour des résultats plus généraux mais moins élémentaires, cf. Guseva, Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1069–1072; MR **17**, 161; et Nirenberg, Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742.] *J. L. Lions* (Nancy).

Bognár, J. Calculation of heat transmission of the floor of a heated room built on the ground. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 409–424 (1955). (Hungarian. Russian and English summaries)

Let R be either a rectangle or an infinite strip in the

(x, y)-plane. The author solves explicitly the following potential problem:

$$T_{xx}+T_{yy}+T_{zz}=0 \quad (z<0);$$
$$T(x, y, 0)+kT_z(x, y, 0)=T_0 \text{ if } (x, y) \text{ is in } R,$$
$$=T_1 \text{ if } (x, y) \text{ is outside } R$$

(stationary temperature distribution), and computes the heat-flow through R. *A. Erdélyi* (Pasadena, Calif.).

Schopf, Andreas. Sur une méthode aux différences pour l'opérateur $\Delta\Delta$. C. R. Acad. Sci. Paris **242** (1956), 1674–1677.

It is well-known [L. Collatz, Deutsche Math. 3 (1938), 200–212; R. Courant, Bull. Amer. Math. Soc. **49** (1943), 1–23, p. 15; MR **4**, 200; G. Pólya, C. R. Acad. Sci. Paris **235** (1952), 995–997; MR **14**, 656] that by inserting in the minimax principle piecewise linear functions broken along the mesh lines of a triangulated square grid one obtains a finite-difference problem giving upper bounds for the eigenvalues of the vibrating-membrane problem $\Delta u+\lambda u=0$.

The author derives the analogous result for the vibrating clamped plate $\Delta\Delta u-\lambda u=0$ with partly clamped and partly simply supported boundary. The trial function must here be continuously differentiable. It is taken as a sum of translates of a standard function in the x and y directions by integral multiples of the mesh width h. The standard function vanishes outside a square of side $3h$ and is taken as the product of a piecewise quadratic function of x and the same function of y. Various integrals involving this function which are needed to find the finite-difference equation giving upper bounds for the eigenvalues are computed. It is pointed out that the given standard function can also be used in variational principles connected with certain static plate problems.

Another possible standard function is the patch function given by J. L. Synge [Convegno Internazionale sulle Equazionialle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Rome, 1955, pp. 39–53; MR **17**, 373], in connection with a biharmonic boundary-value problem. It also vanishes outside a square of side $3h$, but is piecewise quadratic in both variables. *H. F. Weinberger.*

Courant, R.; and Lax, P. Cauchy's problem for nonlinear hyperbolic differential equations in two independent variables. Ann. Mat. Pura Appl. (4) **40** (1955), 161–166.

Solution du problème de Cauchy $(U(x, 0)=0)$ pour le système:

$$(1) \qquad U_t+A(x, t, U)U_x+B(x, t, U)=0$$

(U et B vecteurs, A matrice) où A et B possèdent des dérivées premières continues Lipshitziennes pour t et $|U|$ assez petits. Si A est hyperbolique (vecteurs propres réels $L_t(U)$ correspondant à des valeurs propres distinctes $\lambda_t(U)$ il existe une solution à dérivées premières continues Lipshitziennes: la démonstration donnée ici est une simplification des démonstrations précédentes.

Utilisant les courbes caractéristiques $dx/dt=\lambda_t(U)$, les auteurs écrivent le système (1) sous la forme:

$$\frac{d}{di}(L_t(U)U)=b_t(U)-U\frac{d}{di}L_t$$

où $b_t=-L_tB$ et $d/di=\partial/\partial t+\lambda_t\partial/\partial x$ et construisent sa solution comme point fixe de la transformation $U=T(V)$, V fonction à dérivées premières continues Lipshitziennes, nulle pour $t=0$, U fonction de la même classe définie par la résolution de:

$$\frac{d}{dt}(L_t(V)U)=b_t(V)-V\frac{d}{di}L_t(V).$$

La transformation T est prise d'abord avec pour domaine des vecteurs très réguliers. Ses propriétés (en particulier de contraction, pour la norme maximum, des vecteurs admissibles tels que $V^2+V^2_x+V^2_t\leq R$) permet d'étendre son domaine aux vecteurs à dérivées premières continues Lipshitziennes. T admet un point fixe unique, obtenu par itération, qui donne la solution cherchée de (1).

Le procédé peut s'étendre à des coefficients seulement continus Lipshitziens, donnant alors une solution faible. *Y. Fourès-Bruhat* (Marseille).

Duff, G. F. D. Uniqueness in boundary value problems for the second order hyperbolic equation. Canad. J. Math. **8** (1956), 86–96.

Démonstration d'un théorème d'unicité pour l'équation:

$$a^{ik}\frac{\partial^2 u}{\partial x^i\partial x^k}+b^i\frac{\partial u}{\partial x^i}+cu=0 \quad (i, k=1, \cdots, N),$$

où les coefficients sont des fonctions quatre fois continuement différentiables des x^i, dans un domaine limité par une partie spatiale S, portant des données de Cauchy, et une partie cylindrique T, portant des données du type de Dirichlet, Neumann ou Robin [dans le cas Dirichlet ce théorème a été démontré par une méthode différente par Hörmander, Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, Lunds Univ. Mat. Inst., 1954, pp. 105–115; MR **16**, 483].

Soit C_0 l'intersection de S et T, C_t la surface à $N-2$ dimensions géodésiquement parallèle à C_0 dans T, S_t le domaine à $N-1$ dimensions obtenu en traçant les arcs de géodésiques issus de C_t, normaux à T, de longueur n_0. L'auteur pose:

$$E(t)=\int_{S_t}\{u_t^2+(\nabla_t u)^2\}dS_t$$

et par dérivation, utilisation de la formule de Gauss, et de majorations, il obtient le lemme: si $J(t)=\int_0^t d\tau\int_{C_\tau} u_t u_n dS_\tau \leq 0$ pour $0\leq t\leq t_1$ alors $u\equiv 0$ dans la région couverte par S_t pour $0\leq t\leq t_1$. Il en déduit l'unicité globale par un argument du type Borel-Lebesgue (et l'unicité classique du problème de Cauchy), pour des données de Dirichlet ou Neumann, puis pour des données de Robin en montrant qu'on peut ramener ce cas à celui de Neumann par changement d'inconnue.

La méthode s'étend aux systèmes où les coefficients des termes du second ordre sont les mêmes. *Y. Fourès-Bruhat* (Marseille).

Èidel'man, S. D. Theorems of the type of Liouville's theorem for parabolic and elliptic systems. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 681–684; errata, **101**, 397 (1955). (Russian)

Let U be a regular solution of the parabolic system

$$\frac{\partial U}{\partial t}=\Phi^{2b}\left(\frac{\partial}{\partial x}\right)U$$

defined in the half space $t\leq T$, Φ^{2b} homogeneous of order $2b$. If U satisfies a growth condition, $|U(x, t)|\leq M(r(x)^\alpha+1)$, then U is a polynomial of degree $[\alpha]$ in the space variables $x_1, x_2, \cdots, x_n$ and $[\alpha/2b]$ in time. Naturally this also applies to solutions of elliptic systems. In the case of a single equation

$$\frac{\partial u}{\partial t}=\sum_{k\leq b}\Phi^{2k}(t,\frac{\partial}{\partial x})u,$$

where each operator Φ^{2k} is homogeneous elliptic of order $2k$, Re $\Phi^0\leq-\delta_0<0$, if a solution regular in the half-space $t\leq T$, satisfies the inequality $|u(x,t)|\leq M\exp((\delta_0-\varepsilon)(-t))$ for $\varepsilon>0$, then u is identically zero. The above theorems follow without very great difficulty from Ladyženskaya's estimates [Mat. Sb. N.S. **27(69)** (1950), 175–184; MR **12**, 709], extended to systems by the author [ibid. **33(75)** (1953), 359–382; MR **15**, 712]. *A. N. Milgram.*

Èidel'man, S. D. On fundamental solutions of parabolic systems. Mat. Sb. N.S. **38(80)** (1956), 51–92. (Russian)

Soit $x\in R^n$, t=variable de temps; soit $U=(u_1,\cdots,u_N)$ une fonction définie dans l'espace (x,t), à valeurs dans C^N, solution du système $\partial U/\partial t=P(t,x,(2\pi i)^{-1}\partial/\partial x)U+f(x,t)$, où P est une matrice formée d'opérateurs différentiels en x, à coefficients dépendant de x et de t; on suppose le système parabolique au sens de Petrowsky [Bull. Univ. d'Etat Moscou. Sér. Internat. Sect. A. **1** (1938), no 7; cf. aussi l'A., Mat. Sb. N.S. **33(75)** (1953), 359–382; MR **15**, 712].

§ 1. L'A. suppose que P est à coefficients indépendants de t. Il résout le problème de Cauchy à l'aide de la transformation de Fourier en x; à l'aide d'une part d'évaluations de Petrowsky et de l'A. (loc. cit.), d'autre part des résultats de Gel'fand et Šilov [Uspehi Mat. Nauk (N.S.), **8** (1953), no. 6(58), 3–54; MR **15**, 867], il donne des évaluations précises de la fonction de Green et donne des résultats du type Liouville (cf. aussi l'A., loc. cit., et l'oeuvre analysée ci-dessus).

§ 2. L'A. étudie le cas général: P est à coefficients dépendant de x et de t; à l'aide du § 1 et de la méthode classique d'E. E. Levi, l'A. démontre l'existence d'un noyau élémentaire et étudie la régularité de ce noyau de façon extrêmement précise en fonction de la régularité des coefficients. L'A. applique ce théorème 1) au problème de Cauchy, 2) à la régularité des solutions (hypoellipticité des systèmes paraboliques; ce théorème, dans le cas des coefficients indéfiniment différentiables, a été démontré indépendamment par S. Mizohata [C.R. Acad. Sci. Paris **241** (1955), 1547–1549; MR **17**, 495] par une méthode directe plus simple).

§ 3. Étude (difficile) du prolongement analytique en x_1 du noyau construit au § 2, les coefficients de P étant prolongeables analytiquement en x_1 dans un domaine complexe. § 4. Conséquences du § 3 sur l'analyticité en x_1 des solutions du système. Le § 5 étudie brièvement le cas d'un opérateur donné dans un domaine cylindrique de l'espace (x,t). *J. L. Lions* (Nancy).

Èidel'man, S. D. On Cauchy's problem for parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 913–915. (Russian)

The parabolic system of equations

$$\frac{\partial u_i}{\partial t}=\sum_{j=1}^{N}\sum_{\Sigma K_s\leq 2b}A_{ij}^{(K_1,\cdots,K_n)}(x,t)\,\frac{\partial^{K_1+\cdots+K_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\cdot u_j,\qquad i=1,2,\cdots,N,$$

with coefficients depending both on time and space coordinates is shown to possess a fundamental matrix solution or Green's matrix $Z(t,\tau,x,\xi)$. Estimates of Z and its derivatives up to order $2b$ are shown to have the same general form as in the case of operators with constant coefficients. As a consequence

$$U(x,t)=\int Z(t,t_0,x,\xi)\,\phi(\xi)\,d\xi$$

is a solution with initial value $U(x,t_0)=\phi(x)$, for all functions ϕ for which $|\phi(x)|\leq\exp(k|x|^\beta)$; the range of t depends on k. The solutions generated in this way have a growth rate not exceeding $C_1\exp(C|x|^\beta)$, where $\beta=2b/(2b-1)$ and solutions with such growth are shown to be uniquely determined by their initial values. The author first establishes the existence of the fundamental solution for an equation whose coefficients depend only on t and uses the theorems of Gelfand and Šilov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867] to obtain estimates. Next the parametrix method of E. E. Levi is applied to derive an integral equation for the required Green's matrix. *A. N. Milgram* (Minneapolis, Minn.).

Slobodeckiĭ, L. N. On Cauchy's problem for nonhomogeneous parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 805–808. (Russian)

This note is principally concerned with the extension of Eidelman's results [see the preceding review] to nonhomogeneous parabolic systems, $\partial u/\partial t=A(t,x,\partial/\partial x)u+f(x,t)$, where A is a strongly elliptic matrix operator and $f(x,t)$ a given vector function. By replacing point estimates by mean estimates, e.g.,

$$\int\int|\phi(x)|\exp(-\mu|x|^\sigma)dx<\infty,\text{ where }\sigma=2p/2p-1$$

and $2p$ is the order of A, one obtains, as might be expected, a generalization of Eidelman's existence and uniqueness theorem. The solution of the Cauchy problem is shown to depend continuously, in a sense clearly defined in the note, on the initial data $\phi(x)$, and $f(x,t)$. Proofs are not even indicated, and frequent assumption that the coefficients of the operator depend only on t creates uncertainty as to which theorems the author claims to hold without the restriction. *A. N. Milgram.*

Zagorskiĭ, T. Ya. Some boundary problems for parabolic systems in a half-space. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 11–14. (Russian)

Given the parabolic system (*) $\partial u/\partial t=L(\partial/\partial x)u$, where

$$L\Big(\frac{\partial}{\partial x}\Big)=\Big\|\sum_{k_1+\cdots+k_n=s}a_{ij}^{(k_1\cdots k_n)}\frac{\partial^s}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\Big\|,\quad u=\begin{Vmatrix}u_1\\ \vdots\\ u_N\end{Vmatrix},$$

and $a_{ij}^{(k_1\cdots k_n)}$ is a constant ($i,j=1,\cdots,N$). In the region $-\infty<x_j<\infty$ ($j=1,\cdots,n-1$), $x_n>0$, $t>0$, a solution u is sought which satisfies the conditions:

$\lim_{x_n\to+0}B(\partial/\partial x)u=f(x',t)$, $u|_{t=0}=0$, where

$$B\Big(\frac{\partial}{\partial x}\Big)=\Big\|\sum_{k_1+\cdots+k_n=s_i}b_{ij}^{(k_1\cdots k_n)}\frac{\partial^{s_i}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\Big\|,\quad f(x',t)=\begin{Vmatrix}f_1(x',t)\\ \vdots\\ f_{sN/2}(x',t)\end{Vmatrix}$$

and $b_{ij}^{(k_1\cdots k_n)}$ is a constant ($i=1,\cdots,sN/2$; $j=1,\cdots,N$), $s_i<s$, and $x'=(x_1,\cdots,x_{n-1})$. Let u be the Fourier transform in $(x_1,\cdots,x_{n-1})$ of a function $\tilde u$, and $\tilde{\tilde u}$ the Laplace transform in t of $\tilde u$. Then from (*) is obtained a system (**) of ordinary differential equations in $\tilde{\tilde u}$. The characteristic equation of this system is studied, and by imposing a condition (too long to be described in detail) on the a_{ij}'s, the b_{ij}'s and the solutions of the characteristic equation, a solution $\tilde{\tilde u}$ for (**) is obtained. Applying the inverse Laplace and Fourier transforms to $\tilde{\tilde u}$ yields a solution u of (*) which can be expressed in terms of a

matrix Green's function. The author states that by using an argument similar to that used by O. A. Ladyženskaya [Mat. Sb. N.S. **27(69)** (1950), 175–184; MR **12**, 709] a set of estimates can be obtained for the elements of the matrix Green's function and their derivatives and that the formal steps described above for obtaining solution u can be justified by these estimates. *J. Cronin.*

Zolotarev, G. N. On uniqueness of solution of Cauchy's problem for the heat-conduction equation. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 349–351. (Russian)

Le problème de Cauchy pour l'opérateur $\partial/\partial t-\partial^2/\partial x^2$ équivaut (au moins formellement) à trouver $V(s, t)$, solution de

$$(1)\qquad V(s, t)=-4\pi^2 s^2 V(s, t),\ V(s, 0)=V_0(s) \text{ donné.}$$

On cherche donc une solution de (1) telle que $t\to V(\cdot, t)$ soit une application continue de $t\geqq 0$ dans un espace E de distributions ou de fonctionelles analytiques [cf. Schwartz, Ann. Inst. Fourier, Grenoble **2** (1950), 19–49; MR **13**, 242; Gel'fand et Šilov Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867]. On prend ici E=dual de Z^Φ [où Z^Φ défini comme Z^Ω dans Gurevič, Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 893–895; MR **16**, 720]. La condition nécessaire et suffisante pour que le problème (1) admette une solution unique est que l'intégrale suivante diverge:

$$\int_1^\infty r^{-2}\Phi(r^4)dr.$$

La démonstration utilise [comme le travail initial de Tihonov sur ce sujet, Mat. Sb. **42** (1935), 199–216] la théorie de la quasi analyticité. *J. L. Lions* (Nancy).

Szilvay, Frau G. Berechnung der Temperaturverteilung im Doppelrohrwärmeaustauscher. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 265–274 (1955). (Hungarian. Russian and German summaries)

The author determines axially symmetric solutions of $a^2\Delta u-v\cdot\operatorname{grad} u=0$, u_1 for $0\leqq r\leqq\alpha$, u_2 for $\alpha\leqq r\leqq\beta$, satisfying at $r=\alpha$ the boundary conditions

$$\frac{\partial u_1}{\partial r}=\frac{\partial u_2}{\partial r}=\frac{u_2-u_1}{k}.$$

A. Erdélyi (Pasadena, Calif.).

Czipszer, J. Sur la propagation de la chaleur dans une barre infinie. I. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 395–408 (1955). (Hungarian. Russian and French summaries)

Let S be the strip $-\infty<x<\infty$, $0<t<c$. The author shows that for a solution of $u_{xx}=u_t$ which is continuous in S,

$$\sup_{0<t<h}\int_{-\infty}^{\infty}|u(x, t)|\exp\Big[-\frac{x^2}{4(h-t)}\Big]dx<\infty$$

for each h, $0<h<c$, if and only if

$$u(x, t)=\tfrac{1}{2}(\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp\Big[-\frac{(x-y)^2}{4t}\Big]d\alpha(y),$$

where $\alpha(y)$ is of bounded variation on every finite interval, and the integral is absolutely convergent in S. He also proves that under these circumstances, for each real a and b,

$$2\lim_{t\to+0}\int_a^b u(x, t)dx=[\alpha(b+0)+\alpha(b-0)]-[\alpha(a+0)+\alpha(a-0)],$$

$$|u(x, t)|\leqq\frac{C(h)}{\sqrt{t}}\exp\Big[\frac{x^2}{4(h-t)}\Big]$$

for $0<t<h<c$. The author states that he does not know whether the last inequality characterizes the class of solutions of the heat equation investigated here.

A. Erdélyi (Pasadena, Calif.).

Freud, G. Über Wärmeleitungs- und Diffusionsprobleme mit zusammengesetzten Randbedingungen. I. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 369–394 (1955). (Hungarian. Russian and German summaries)

The author considers problems of heat conduction in a system consisting of a bar and a concentrated body. Accordingly, he has two unknown functions, $V(x, t)$, the temperature of the bar at the point x and at time t, and $U(t)$, the temperature of the body at time t. A typical problem is the following one: $x\geqq 0$, $t\geqq 0$, $V_{xx}=aV_t$, $V(x, 0)=0$, $U(0)=U_0$, $V_x(0, t)=C[U(t)-V(0, t)]$, $U'(t)+B[U(t)-V(0, t)]=Q(t)$. Here $Q(t)$ is a given function, representing the heat fed into the body. He solves several problems of this nature. He also proves the uniqueness of the solution of this kind of problems. *A. Erdélyi.*

Karol', I. L. On the theory of boundary problems for an equation of mixed elliptic-hyperbolic type. Mat. Sb. N.S. **38(80)** (1956), 261–282. (Russian)

Soit dans le plan des (x, y) l'opérateur différentiel S_α défini par $S_\alpha u=u_{xx}+yu_{yy}+\alpha u_y$, α=constante réelle. Soit Ω l'ouvert limité par les caractéristiques AC, BC issues de $A=(0, 0)$, $B=(1, 0)$ et par Γ courbe dans $y>0$ joignant A et B. Soit Ω_+ (resp. Ω_-) la partie de Ω située dans $y>0$ (resp. $y<0$). Problème M (resp. E): trouver u dans Ω solution de $S_\alpha u=0$, prenant des valeurs données sur la frontière de Ω (resp. sur la seule portion Γ de la frontière) et assujettie à des conditions de régularité convenables (l'A. considère pour E deux sortes de régularité: problèmes E_1 et E_2). Théorème I: Si $\alpha\leqq 0$, le problème M admet une solution unique. Théorème II: Si $\alpha>0$, le problème E admet au plus une solution. Il admet une solution unique si $0<\alpha<1$ (ou $0<\alpha\leqq 1$, selon les conditions de régularité). Le principe de la méthode est simple: on commence par résoudre le problème de Goursat dans Ω_+ si $\alpha<0$ (resp. de Dirichlet dans Ω_+ si $\alpha>0$) et ensuite on résout Dirichlet ou Cauchy. Pour le problème de Dirichlet dans Ω_+, on utilise essentiellement M. V. Keldyš, Dokl. Akad. Nauk SSSR (N.S.) **77** (1951), 181–183 [MR **13**, 41]. *J. L. Lions* (Nancy).

Karol', I. L. On boundary problems for equations of mixed type. Vestnik Leningrad. Univ. **11** (1956), no. 1, 177–181. (Russian)

Soit dans le plan des (x, y) l'opérateur différentiel S_α défini par $S_\alpha u=u_{xx}+yu_{yy}+\alpha u_y$, α=constante réelle. Soit AC et BC les caractéristiques issues de $A=(0, 0)$ et $B=(0, 1)$ dans $y<0$; soit Γ_- une courbe régulière joignant A à B dans $y<0$ et située à l'intérieur du triangle curviligne ABC; soit Γ_+ une courbe régulière joignant A à B dans $y>0$; soit Ω l'ouvert limité par Γ_- et Γ_+. On cherche u solution de $S_\alpha u=0$ dans Ω, u étant donnée sur Γ_+ et u étant donnée ainsi que ses dérivées du premier ordre sur

Γ_- (avec des hypothèses de régularité convenables). Si $\alpha \leq 0$, ce problème admet une solution unique (même méthode que dans l'analyse précédente). Etude de la régularité de la solution. *J. L. Lions* (Nancy).

Agmon, S. Boundary value problems for equations of mixed type. Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 54–68. Edizioni Cremonese, Roma, 1955. 3000 Lire.

The author presents a proof without details of the existence of solutions for two boundary-value problems for mixed equations of the form $zu_{xx}+u_{zz}+a(x,z)u=b(x,z)$ in a domain D bounded by a smooth arc L in the elliptic region and two characteristics in the hyperbolic region. On L and one of the characteristics Γ_B, u (or its conormal derivative) is prescribed. Weaker existence proofs for the first problem have been given in detail by Germain and Bader, and outlined by Protter.

For the Tricomi equation, $a=0$, it is shown, using the fundamental solution of Germain and Bader, that, if the prescribed values are continuous on $L+\Gamma_B$ and twice Hölder continuously differentiable on Γ_B, then a strong solution exists in both cases. Existence for $a \not\equiv 0$ can then be established if a suitable uniqueness holds, if the same problem admits a strong solution for the Tricomi equation and if a and b are regular enough. For the Tricomi equation, the solution is found by solving a pair of singular integral equations that hold on L and Γ_B. For the general problem the solution is found using a number of linear transformations of the class of continuous functions in $\bar{D}$ considered as a Banach space. The chief difficulty lies in showing that each of these transformations is completely continuous or can be approximated appropriately by a completely continuous transformation.

The author adds that he can now extend the results to domains bounded by non-characteristic arcs in the hyperbolic region. *C. S. Morawetz* (New York, N.Y.).

Višik, M. I. Mixed boundary-value problems for equations having a first derivative in the time and an approximate method for their solution. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 189–192. (Russian)

In the cylindrical domain $Q=D\times(0<t<1)$ where D is a bounded region of Euclidean n-space, with boundary Γ, the system of differential equations

$$Lu \equiv A\left(x,t,\frac{\partial}{\partial x_i}\right)\frac{\partial u}{\partial t}+B\left(x,t\,\frac{\partial}{\partial x_i}\right)u=h(x,t)$$

are considered, where A_r and B_s are differential operators of order r and s respectively and u and h are vector functions. The author treats the cases: a) where $r=2m\geq s$ and A_{2m} is a strongly elliptic positive operator, B_s is arbitrary; b) $r=2m<2k=s$, A_{2m} is a self-adjoint strongly elliptic positive operator and B_{2k} is a general strongly elliptic operator; and c) $r=2m<s=2k$, A_{2m} is a general strongly elliptic operator, (positive), and B_{2k} is a strongly elliptic operator admitting a representation $B_{2k}=B'+B''$, where B' is a self-adjoint strongly elliptic operator and B'' is an operator of order $\leq 2m$. The theorems are too long to be formulated here. In each of the above cases the author exhibits the existence and uniqueness of weak solutions $u(x,t)$ having prescribed initial value $u(x,0)=\phi(x)$ at $t=0$ and with u and its first $j-1$ normal derivatives vanishing on the boundary $\Gamma\times(0<t<1)$, $j=\max(m,k)$. The problem of when the weak solutions are regular is not discussed. Writing $\tilde{u}_n=\sum c_i(t)u_i(x)$, the relation $(L\tilde{u}_n, u_i)=(h, u_i)$ determines a system of ordinary differential equations for the functions $c_i(t)$. Choosing initial values $c_i(0)$ so that $\sum c_i(0)u_i(x)$ approximates $\phi(x)$ it is shown that the functions $\tilde{u}_n$ converge weakly to the solutions previously discussed. The author calls the latter approximation method the method of Galerkin.

A. N. Milgram (Minneapolis, Minn.).

Višik, M. I. Mixed boundary problems and an approximate method for their solution. Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 193–196. (Russian)

The system of partial differential equations $Lu\equiv u_t+A(t)u=h$, where u, h denote vector functions, is said to be of parabolic type if $A(t)$, the spatial part of L, is an operator of order $2m$ which satisfies strong elliptic conditions uniformly in the domain $Q=D\times(0<t<T)$, where D is a bounded domain in Euclidean n-space. The author establishes existence and uniqueness of generalized weak solutions u to such equations, where u and its first $m-1$ normal derivatives vanish on the boundary of D for each $t<0$, and $u(x,0)=\phi(x)$ is a given initial vector function. The coefficients occurring in L are assumed to have a derivative with respect to t which is bounded in Q, and $h \in L_{p,2}(Q)$. The method of proof, in essence, consists in forming a Hilbert space with norm $(v(x,0), v(x,0))+\int(v, Av)dt$ from a pre-Hilbert space of functions of the form $v=\int w(x,\mu)d\mu$ for $t<T$, $v\equiv 0$ for $t\leq T$ and ultimately applying the Fréchet-Riesz theorem. No indication is given as to whether the weak solutions are regular. The method of Galerkin (see preceding review) yields a sequence of functions which converge weakly to the solution.

A. N. Milgram (Minneapolis, Minn.).

Višik, M. I. Mixed boundary problems for systems of differential equations having a second derivative with respect to time, and an approximate method for their solution. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 409–412. (Russian)

In this note the author discusses systems of differential equations of the form

$$Lu\equiv A_{m'}\left(x,t,\frac{\partial}{\partial x_i}\right)\frac{\partial^2 u}{\partial t^2}+B_{r'}\left(x,t,\frac{\partial}{\partial x_i}\right)\frac{\partial u}{\partial t}+C_{s'}\left(x,t,\frac{\partial}{\partial x_i}\right)u=h(x,t).$$

Various cases are treated, depending on assumptions of ellipticity and self-adjointness of the operators involved. To illustrate: Let $m'=2m$, $r'=2r>2m$, $s'\leq 2r$ and let A_{2m} be a self-adjoint strongly elliptic positive operator. Let $B_{2r}=B'+B''$ where B' is a strongly elliptic positive operator and B'' is arbitrary of order $\leq 2m$. Finally, let $C_{s'}$ be an arbitrary operator of order $2r$. Then there exists a unique weak solution $u(x,t)$ with initial values $u(x,0)=\phi(x)\in W_2^{\varrho}(0)$ and $u_t(x,0)=\psi(x)\in W_2^m(0)$ and having boundary values u and its first $m-1$ normal derivatives zero on $\Gamma\times(0<t<1)$. The solution depends continuously on h, ϕ, ψ. In addition to such theorems, the author shows u admits generalized derivatives with respect to t up to order $\varrho+2$ when the coefficients and ϕ, ψ and h admit sufficiently many derivatives. From this, applying theorems of Sobolev and Friedrichs it is deduced that the generalized solutions are regular. As in the papers reviewed above, the method of Galerkin is applied to obtain a sequence of functions which in this case converge in the strong sense in $R_0^m(Q)$ to the exact solution $u(x,t)$.

A. N. Milgram (Minneapolis, Minn.).

Maurin, K. Über gemischte Rand- und Anfangswertprobleme im Grossen für eine Klasse von Gleichungssystemen auf differenzierbaren Mannigfaltigkeiten. Eine Begründung der Fouriermethode. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 471–475.

Let A be a self-adjoint operator on a Hilbert space H which is bounded from below and has the spectral resolution $A=\int \lambda dE(\lambda)$. Then the solution u of the equation $u_{tt}=Au$, $u(0)=f \in D(A)$, $u_t(0)=g \in D(\sqrt{A})$ ($u=u(t) \in H$, $t\geqq 0$, the index t denotes derivation) is given by

$$u=\int \cos \lambda^{\frac{1}{2}}t \cdot dE(\lambda)f+\int \sin \lambda^{\frac{1}{2}}t\cdot\lambda^{-\frac{1}{2}}dE(\lambda)g,$$

it is unique and depends continuously on f and g. When A is an extension of an elliptic differential operator of order s with sufficiently regular coefficients defined on a differentiable manifold Ω, and H is the space of square-integrable functions on Ω with values in a finite-dimensional Hilbert space, it is shown that $u \in C^s(\Omega)$ for every t provided that $f \in D(A^m)$, $g \in D(A^{m-\frac{1}{2}})$ and $m>n(2s)^{-1}+1$. If $f \in D(A^{m+1})$ and $g \in D(A^m)$, then u is also in $C^s(\Omega)\times C^2(T)$, where T is the interval $t\geqq 0$. This generalizes a result of Yosida [Proc. Japan. Acad. **28** (1952), 396–403; MR **14**, 757]. A simple proof of a similar theorem by Browder [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 459–463; MR **16**, 134] concerning parabolic equations is also given. *L. Gårding* (Lund).

Colino, Antonio. Application of the Wiener-Hopf technique to diffraction problems. Rev. Ci. Apl. **9** (1955), 481–494. (Spanish)

Iskenderov, Š. A. On the generalized Cauchy problem for a system. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 106–127. (Russian. Azerbaijani summary)

The author considers the Cauchy problem for the system (*):

$$\frac{\partial^m u_l}{\partial t_1 \partial t_2 \cdots \partial t_m}=\sum_{j=1}^{N}\sum_{(k_\nu)} A_{lj}^{(k_1,\cdots,k_n)}(t_1,\cdots,t_m)\frac{\partial^{k_1+\cdots+k_n}u_j}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}+f_l(t_1,\cdots,t_m;x_1,\cdots,x_n)$$

with initial conditions:

$$u_l|_{t_\nu=t_\nu^0}=\phi_{l\nu}(x_1,\cdots,x_n),$$

where $i=1, 2, \cdots, N$; $\nu=1, 2, \cdots, m$; the sum $\sum_{(k_\nu)}$ is taken over all derivatives with respect to the x_l not exceeding a total order M, and the functions A_{lj}, f_l, and $\phi_{l\nu}$ are suitably well-behaved complex functions of real arguments.

Such a problem is "well-set" in the usual sense if the solution functions exist, are uniquely determined, and depend continuously on the initial data. In contrast to this, the author, following Petrovsky, introduces the concept of a "uniformly well-set problem," which is defined as follows:

The system (*) together with the initial values $\phi_{l\nu}$ constitute a "uniformly well-set problem" for $0\leqq t_s^0\leqq T_s$ if: (1) for every set of functions $\phi_{l\nu}(x_1, x_2, \cdots, x_n)$, bounded, together with their partial derivatives up to some order L, on all $(x_1, \cdots, x_n)$ hyperplanes, there exists a unique set of functions u_l, bounded together with their relevant derivatives, which, for all t_s in $0\leqq t_s^0<t_s\leqq T_s$ satisfy the system (*) and, for $t_s=t_s^0$, take on the initial values ϕ_{ls}; (2) for every t_s^0 in $0\leqq t_s\leqq T_s$, and for an arbitrary fixed $\varepsilon>0$ there exists an η, independent of t_s^0, such that, whenever $\phi_{l\nu}$ and their derivatives up to order L change by less than η, the functions u_l on $t_s^0\leqq t_s\leqq T_s$ change by less than ε. The distinctive feature of this definition, which justifies the name "uniform", is that η is independent of t_s^0.

The author introduces "condition A": define $v_l^{(l)}$ by

$$\frac{\partial^m v_l^{(l)}}{\partial t_1\partial t_2\cdots\partial t_m}=\sum_{j=1}^{N}\sum_{(k_\nu)} A_{lj}^{(k_1,\cdots,k_n)}(t_1,\cdots,t_m)(\sqrt{(-1)}\beta_1)^{k_1}\cdots(\sqrt{(-1)}\beta_n)^{k_n}v_j^{(l)}$$

with initial conditions

$$v^{(l)}\Big|_{t_\nu=t_\nu^0}=\begin{cases}0, & i\neq l\\ 1, & i=l.\end{cases}$$

Then "condition A" is said to be satisfied if there exist $C>0$, $P>0$ such that the $v_l^{(l)}$ satisfy

$$|v_l^{(l)}|<C[1+\beta_M]^P$$

as $\beta_M\equiv \max|\beta_s|\to\infty$, β_s real.

The main result of the paper is that the system (*), with the initial conditions stated, is a uniformly well-set problem in $(0, T_s)$ if and only if condition A is satisfied in $(0, T_s)$. The proof of the necessity of condition A is based on the use of an initial-value function

$$\phi=\varepsilon\exp(\sqrt{(-1)}\textstyle\sum \beta_k x_k);$$

the proof of sufficiency employs Fourier transforms (with respect to the x_l), and involves extensive estimation of the rate of growth of certain integrals. *R. B. Davis.*

See also: Schläfli, p. 814; Harrik, p. 830; Smirnow, p. 833; Bers, p. 837; Szilvay und Zergényi, p. 845; Germain, p. 846; Marčenko, p. 853; Ehrenpreis, p. 877; Mihlin, p. 880; Roma, p. 901; Tersenov, p. 917.

Difference Equations, Special Functional Equations

Fort, Tomlinson. Linear difference equations and the Dirichlet series transform. Amer. Math. Monthly **62** (1955), 641–645.

The author develops a method of solving linear difference equations with constant coefficients by means of a Dirichlet-series transform technique analogous to the familiar method employing the Laplace transform. Given $a(t)$ defined for integral values of t, the Dirichlet-series transform of $a(t)$ is defined to be

$$f(s)=D\{a(t)\}=\sum_{t=0}^{\infty} m^{-st}a(t),\quad m>1,\ s>1,$$

where it is assumed that the series converges. By summation by parts techniques and manipulation of series the author determines the Dirichlet transforms for $\Delta^n a(t)$ and $a(t+n)$ and their relation to the transform of $a(t)$. A short table of transforms for the readily encountered elementary functions is prepared and the results applied to the solution of a typical problem. *P. E. Guenther.*

Bellman, Richard. A functional equation arising in allocation theory. J. Soc. Indust. Appl. Math. **3** (1955), 129–132.

The functional equation

$$f(x)=\max_{0\leqq y(x)\leqq x}\{g[y(x)]+h(x-y(x))+f[ay(x)+b(x-y(x))]\}$$

is examined for concave g, h under certain further restrictive conditions. The proofs are sketched.

J. Aczél (Debrecen).

See also: Karmišin, p. 792; Marguet, p. 812. Gyires, p. 830; El'sgol'c,p. 847; Schopf, p. 856; Lumiste, p. 900;

Integral Equations, Equations in Infinitely Many Variables

Fridman, V. M. Method of successive approximations for a Fredholm integral equation of the 1st kind. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 233–234. (Russian)

Let $K(x, s)=K(s, x)$ be a square-summable positive definite kernel and assume the equation

$$\int_a^b K(x, s)\varphi(s)ds=f(x),\ f(x)\in L_2(a, b),$$

is solvable. The author shows that the solution is the limit in the mean of the sequence $\{\varphi_n(x)\}$ defined by

$$\varphi_n(x)=\varphi_{n-1}(x)+\lambda[f(x)-f_{n-1}(x)],$$

where $\varphi_0(x)\in L_2(a, b)$, $f_{n-1}(x)=\int_a^b K(x, s)\varphi_{n-1}(s)ds$, $0<\lambda<2\lambda_1$, and λ_1 is the least eigenvalue of $K(x, s)$.

J. V. Wehausen (Providence, R.I.).

Sumner, D. B. Abel's integral equation as a convolution transform. Proc. Amer. Math. Soc. **7** (1956), 82–86.

Abel's integral equation $F(X)=\int_0^X (X-T)^{-\alpha}\Phi(T)dT$ $(0<\alpha<1,\ X>0)$ can be written in the form

$$(*)\qquad f(x)=\int_{\mathbf{X}}^{\infty}(e^{t-x}-1)^{-\alpha}\phi(t)dt.$$

For $f(x)$ so defined, let

$$\pi^{-1}\sin\pi\alpha\int_{-\infty}^{\infty} f(x-v)(1+e^{-v})^{-1}dv=g(x);$$

then $g(x)=\int_{-\infty}^{\infty}(1+e^{t-x})^{-\alpha}\phi(t)dt$ and

$$\tfrac{1}{2}[\phi(x+)+\phi(x-)]=(2\pi)^{-1}\lim_{\theta\to\pi}\int_{-\theta}^{\theta}(1+e^{-iy})^{\alpha-1}g'(x+iy)dy.$$

This formula of integro-differential type for the solution of equation (*) is established, under certain conditions on $\phi(t)$, with the aid of properties of a generalized Stieltjes transform. *R. V. Churchill* (Ann Arbor, Mich.).

Jaeckel, K. Herleitung von Umkehrformeln und Identitäten zur Behandlung besonderer Integralgleichungen. Z. Angew. Math. Mech. **35** (1955), 474–475.

Starting from a Fourier series representation and identities, the author obtains, for example,

$$f(\phi)=-\frac{1}{2\pi}\int_0^{\pi} g(\theta)\log\frac{1-\cos(\phi+\theta)}{1-\cos(\phi-\theta)}d\theta,$$

$$g(\theta)=-\frac{1}{2\pi}\int_0^{\pi} f''(\phi)\ln\frac{1-\cos(\psi+\theta)}{1-\cos(\psi-\theta)}d\psi$$

for $f(x)$ in $C^{(2)}(x)$. Application of these results is made to Prandtl's circulation equation, reducing an integro-differential equation to an ordinary integral equation, obtaining known results. *G. E. Latta.*

Vasil'ev, V. V. On the solution of systems of linear homogeneous generalized integro-differential equations. Irkutsk. Gos. Univ. Trudy. **8** (1953), no. 1, 3–8. (Russian)

The author considers systems of linear homogeneous integro-differential equations of the form

$$0=\frac{d}{dx}Z_i(x)+\sum_j a_{ij}(x)Z_j(x)+\lambda\int_a^b\sum_j k_{ij}(x, y)Z_j(y)dy.$$

This system is reduced to a system of integral equation for which a power series solution in λ can be found for sufficiently small λ. He observes that, for a pair of equations, if all a_{ij} are constants and $K_{i2}(x, y)=0$ $(i=1, 2)$ and $k_{11}(x, y)$ are differentiable in x, then the system can be reduced to a single integro-differential equation which can be solved by a method due to Nekrasov (no reference is given). *D. C. Kleinecke* (Albuquerque, N.M.).

Bykov, Ya. V. On a class of integro-differential equations. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 6, 3–6. (Russian. Uzbek summary)

The author gives a method for solving the integro-differential equation

$$L(y)+\int_a^x\sum_{j=1}^{b} P_j(x-t)\cdot\exp\beta_j(t-x)M(y(t))dt=0,$$

where L and M are homogeneous linear differential operators with constant coefficients, $P_j(u)$ are polynomials in u, and β_j are constant. He shows by examples how the method fails when the equation has no solution and when the equation is equivalent to an algebraic equation.

D. C. Kleinecke (Albuquerque, N.M.).

See also: Agmon, p. 859.

Calculus of Variations

★ **Ahiezer, N. I. Lekcii po variacionnomu isčisleniyu. [Lectures on the calculus of variations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 248 pp. 4.40 rubles.

The main distinguishing feature of this excellent short textbook is the attention given to the so-called direct methods of the calculus of variations, in which the existence of minima is proved by compactness and lower semi-continuity arguments. In addition there is some treatment of double integrals, including Haar's theorem. The author intends to "acquaint the student with the leading methods of calculus of variations, classical and direct." There are many worked out examples and methods in both categories. *J. M. Danskin* (Princeton, N.J.).

★ **Lyusternik, L. A. Kratčaĭšie linii. Variacionnye zadači. [Shortest curves. Variational problems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 103 pp. 1.50 rubles.

Elementary popular exposition.

Krzywicki, A.; Rzewuski, J.; Zamorski, J.; and Zięba, A. Non-local problems in the calculus of variations. I. Ann. Polon. Math. **2** (1955), 77–96.

The problems in question are given by

$$W=\int L[t, x_i(t), \dot{x}_i(t)]\,dt+\lambda\iint M[t, \tau, x_i(t), x_i(\tau)\dot{x}_i(t), \dot{x}_i(\tau)]\,dtd\tau$$

in which the variable functions are the n functions $x_1(t), \cdots, x_n(t)$ of the single variable t. Both t and τ vary on the interval (a, b). The paper is the first part of a systematic investigation of their structure. The "Euler equations" turn out to be in integro-differential form. There is an existence theorem for these equations, which are found to be equivalent under general conditions to pure differential equations. The Weierstrass function and the second variation are left to a later paper. The authors state that the restriction to first-order derivatives and double integrals is not essential, but that the restriction to one independent variable is essential.

J. M. Danskin (Princeton, N.J.).

Glansdorff, P. Sur le calcul par récurrence de la variation seconde d'une intégrale multiple. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 124–129.

This note makes some comments on the form of the variations of multiple integrals of the second and higher orders, following the notations of De Donder.

L. M. Graves (Chicago, Ill.).

Auslander, Louis. The use of forms in variational calculations. Pacific J. Math. **5** (1955), 853–859.

The author presents a method involving the differential forms of E. Cartan for the evaluation of the first and second variations of a line integral (in the sense of the calculus of variations). Necessary and sufficient conditions for the vanishing of the first variation are found. The integrand is then assumed to satisfy the requirements of a metric function of an n-dimensional Finsler space, and using Cartan's formalism based on the euclidean connection the second variation along a geodesic is computed. These results are generalizations of corresponding formulae as developed for Riemannian spaces by I. J. Schoenberg [Ann. of Math. (2) **33** (1932), 485–495]. Remark: In connection with an alternative discussion of the relevant geometrical properties of the second variation the paper of E. T. Davies [J. London Math. Soc. **24** (1949), 241–247; MR **11**, 400] should be mentioned. *H. Rund.*

de Castro Brzezicki, A. Extremal curves of the variational problem $\delta \int f(k, k')ds$. Rev. Mat. Hisp.-Amer. (4) **15** (1955), 71–78. (Spanish)

In this paper we study the extremal curves of the integral $\int f(\varkappa, \varkappa')ds$, in which $\varkappa$ indicates the curvature of an arc and $\varkappa' = d\varkappa/ds$. We deduce the differential system which these extrema satisfy and study its solution; here notable reductions are accomplished, bringing the system, originally of the fifth degree, down to one of the second degree. The results obtained have application in the theoretical study of the design of communication routes, in which large velocities must be developed, problems in which it is essential to maintain the intrinsic coordinates of the curves. (From the author's summary.)

J. M. Danskin (Princeton, N.J.).

Pucci, Carlo. Sulla risoluzione di un problema isoperimetrico. Ann. Mat. Pura Appl. (4) **39** (1955), 393–400.

This paper is concerned with determining the shape of a surface of minimum area enclosing a given volume, under the restriction that all the horizontal sections of the surface are similar to a given plane domain of suitably restricted shape. Under these conditions the volume is given by

$$V = \sigma \int_{-h}^{h} [\delta(z)]^2 dz$$

and when the area of the top and bottom sections is neglected the area is given by

$$A = \int_{-h}^{h} \int_{0}^{L} \delta(z)[1 + (\delta'(z)\phi(s) + \alpha'(z)\psi(s))^2]^{\frac{1}{2}} ds dz,$$

where ϕ and ψ are given functions and σ and L are given constants. The author shows the existence of a minimizing pair

$$\alpha_0(z),\ \delta_0(z)\ (-h_0 \leq z \leq h_0),$$

and derives the differential equations they must satisfy. The case when the area of the bottom section is fixed and the area of the variable top section is included in the quantity to be minimized is treated by means of the preceding case. *L. M. Graves* (Chicago, Ill.).

Faedo, Sandro. Esistenza dell'estremo in un problema di calcolo delle variazioni riguardante il funzionamento delle centrali alimentate dalle maree. Ann. Mat. Pura Appl. (4) **40** (1955), 321–347.

This paper considers the problem of the maximum energy available from ocean tides running into and out of a bay under certain conditions. The resulting variational problem is of an unorthodox type. The author shows the existence of a maximum. *L. M. Graves.*

See also: Nikol'skiĭ, p. 855.

Theory of Probability

Kamalov, M. K. Moments and cumulants of quadratic and bilinear forms of normally distributed variables. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 69–77. (Russian)

Let a) $Q = \sum_{h,k=1}^{n} a_{h,k} x_h x_k$ be a positive definite quadratic form, where $x_1, \cdots, x_n$ follow an n-variate normal distribution with zero means; b) $B(x, y) = \sum_{h,k=1}^{n} \lambda_h x_h y_h$, λ_h real, a bilinear form in normal variables x and y with zero means and unit variances, $\operatorname{cor}(x_h, y_h) = \varrho$ (independence for $h \neq k$); c) $\beta(x, y) = \sum_{j=1}^{n} \lambda_j x_j y_j$ with $\operatorname{cor}(x_h, x_k) = r_{hk}$, $\operatorname{cor}(y_h, y_k) = \varrho_{hk}$ (x_h, y_k uncorrelated). The paper derives general formulas for the moments and cumulants of these three forms. The method of derivation is based on characteristic functions and Faà di Bruno's formula. Explicit expressions are given for moments and cumulants of low order. *G. E. Noether.*

Medgyessy, P. Some recent results concerning the decomposition of compound probability distributions. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 155–169 (1955). (Hungarian. Russian and English summaries)

In an earlier paper [same Közl. **2** (1953), 165–177; MR **16**, 270] the author investigated methods for the determination of the components of a mixture of normal distributions. In the first part of the present paper he discusses various approximations for the function $\varphi^*(t)$ [for the notation see the review quoted above] and estimates also their error term. In the second part of the paper the method is modified so as to permit the decomposition of mixtures of stable laws. *E. Lukacs.*

Medgyessy, P. A supplement to the paper "Some recent results concerning the decomposition of compound probability distributions". Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 331–341 (1955). (Hungarian. Russian and English summaries)

This is a supplement to the author's earlier papers [same Közl. **2** (1953), 165–177; MR **16**, 270; and the paper reviewed above], it gives the derivations of the estimates used in the second paper. *E. Lukacs.*

Medgyessy, P. Decomposition of discrete compound probability distributions. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 139–153 (1955). (Hungarian. Russian and English summaries)

The author considers a mixture of binomial (respectively Poisson) distributions with different parameters and gives a method for the determination of its components.

The work originated apparently in connection with a chemical application [Medgyessy, Renyi, Tettamanti, and Vincze, same Közl. 3 (1954), 81–97; MR 17, 873].
E. Lukacs (Washington, D.C.).

Medgyessy, P. An approximate method for the decomposition of a compound of Cauchy frequency functions. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 321–329 (1955). (Hungarian. Russian and English summaries)

Let $g(x)=\sum_{k=1}^{N} A_k\sigma_k\pi^{-1}[\sigma_k^2+(x-m_k)^2]^{-1}$ be a mixture of Cauchy frequency functions. The paper deals with the determination of the components of $g(x)$. Denote by $g^*(x,\lambda)$ the frequency function which is obtained from $g(x)$ by replacing σ_k by $(\sigma_k-\lambda)$ and form $G^*(x,\lambda)=\frac{1}{2}[g^*(x,\lambda)+g^*(x,-\lambda)]$. Then it is possible to expand $G^*(x,\lambda)$ into a power series in λ whose coefficients contain only the even-order derivatives of $g(x)$. If $\lambda\to\sigma_1$ then one component of $G^*(x,\lambda)$ becomes degenerate; this fact is easily recognized in the graph of $G^*(x,\lambda)$ and is used to determine the first component. The author discusses the numerical aspects of the method and in particular the limitations which are imposed by the fact that he uses only a finite segment of the power series for $G^*(x,\lambda)$.
E. Lukacs (Washington, D.C.).

Wheeler, Ruric E. A variable probability distribution function. Ann. Math. Statist. 27 (1956), 196–199.

Consider Bernoulli trials where the probability of success on a single trial depends both on the number of the trial and on the number of previous successes. An expression for the probability of x successes in n trials is given.
M. Muller (Ithaca, N.Y.).

Zinger, A. A. On a problem of A. N. Kolmogorov. Vestnik Leningrad. Univ. 11 (1956), no. 1, 53–56. (Russian)

Let $x_1, \cdots, x_n$ be independent, identically distributed chance variables. Let $n\bar{x}=\sum x_i$, $S^2=\sum(x_i-\bar{x})^2$, $z_i=(x_i-\bar{x})S^{-1}$ $(i=1,\cdots,n)$. The chance vector $Z=(z_1,\cdots,z_n)$ is distributed on the $(n-2)$-dimensional sphere T, defined by $\sum z_i=0$, $\sum z_i^2=1$. If the chance variables are normal the vector, Z is uniformly distributed on T. The author proves that, if $n\geqq 6$, and Z is uniformly distributed on T, the distribution of x_i is normal.
J. Wolfowitz.

Girault, Maurice. Fonction caractéristique de $|X|$ et de lois conditionnelles. C. R. Acad. Sci. Paris 242 (1956), 445–446.

The following two theorems are stated. Proofs are sketched. 1. If $\varphi(t)=\Lambda(t)+iB(t)$ is the characteristic function of a random variable X, then $|X|$ has the following characteristic function

$$\Phi(t)=\Lambda(t)-\frac{i}{\pi}\lim_{\substack{\varepsilon\to 0\\ U\to\infty}}\int_{-U}^{-\varepsilon}\int_{+\varepsilon}^{+U}\left[\frac{\Lambda(t+u)}{u}\,du\right].$$

2. If $\varphi(u,v)$ is the characteristic function of a pair of random variables (X,Y) then the conditional distribution of Y given $X=x$ has the following characteristic function

$$\phi_{x|y}(v)=\frac{\int\varphi(t,v)e^{-itx}dt}{\int\varphi(t,0)e^{-itx}dt}.$$

S. C. Moy (Detroit, Mich.).

Barton, D. E.; and David, F. N. Sums of ordered intervals and distances. Mathematika 2 (1955), 150–159.

From the authors' summary, "A line of unit length is randomly divided by $(n-1)$ points into n intervals. The distances of these points from one end of the line are x_i $(i=1,\cdots,n-1)$. The n intervals g_i $(i=1,2,\cdots,n)$ between successive points, including the end points, are ordered in magnitude: $0\leq g_1\leq g_2\leq\cdots\leq g_n\leq 1$, $\sum_{i=1}^{n} g_i=1$. We derive here the probability distribution functions of $\sum_{i=r}^{n} g_i$ and $\sum_{i=r}^{s} x_i$ $(r<s)$." *M. Muller* (Ithaca, N.Y.).

Badrikian, A. Contribution à la théorie moderne des probabilités. Cahiers Rhodaniens 5 (1953), 1–17.

The author describes some of the concepts and results of a generalized version of ordinary probability theory in the language of Bourbaki. Example: "champ de probabilité" is defined as a pair (E,μ), where E is a locally compact space and μ is a bounded, positive Radon measure in E. The generalization consists in considering random variables with values in a Banach space. There is a brief discussion of random functions, their correlation function, and their integrability. The paper ends with the phrase "à suivre" in what appears to be the middle of a section on stationary random functions on locally compact abelian groups.
P. R. Halmos (Chicago, Ill.).

Hewitt, Edwin; and Savage, Leonard J. Symmetric measures on Cartesian products. Trans. Amer. Math. Soc. 80 (1955), 470–501.

In the following, "probability" and "algebra" can mean either finitely additive probability, Boolean algebra (finite situation) or countably additive probability, Boolean σ-algebra (infinite situation). $\mathcal{X}$: algebra of subsets of a set X with X as unit. $\tilde{X}$: Cartesian product of a countably infinite sequence of replicas of X. $\tilde{\mathcal{X}}$: smallest algebra of subsets of $\tilde{X}$ containing all cylinders. $\mathbf{P}$: set of all probabilities π on $\mathcal{X}$; $\tilde{\mathbf{P}}$: set of the corresponding product measures $\tilde{\pi}$ on $\tilde{\mathcal{X}}$. $\tilde{T}$: arbitrary 1-to-1 mapping of the positive integers onto themselves leaving all but a finite members of integers fixed. For $a=(a_1,\cdots,a_n,\cdots)\in\tilde{X}$, $S_{\tilde{T}}a=(a_{\tilde{T}(1)},\cdots,a_{\tilde{T}(n)},\cdots)$. A probability σ on $\tilde{\mathcal{X}}$ is called "symmetric" if and only if $\sigma(S_{\tilde{T}}{}^{-1}A)=\sigma A$ for all $A\in\tilde{\mathcal{X}}$ and for all $\tilde{T}$. $\tilde{\mathbf{S}}$: set of symmetric probabilities on $\tilde{\mathcal{X}}$. For all real numbers λ and all $E\in\mathcal{X}$, $\mathbf{N}(E,\lambda)=\{\pi;\pi\in\mathbf{P},\pi E\leq\lambda\}$. $\mathcal{P}^*$: smallest Boolean σ-algebra containing all $\mathbf{N}(E,\lambda)$. It is proved (Th. 4.1) that for every $A\in\tilde{\mathcal{X}}$, πA as a function of π is $\mathcal{P}^*$-measurable. A symmetric probability σ is called "presentable" if and only if there exists a countably additive probability μ on $\mathcal{P}^*$ such that $\sigma A=\int_{\mathbf{P}}(\tilde{\pi}A)d\mu(\pi)$. $\tilde{\mathcal{X}}$ is said to be "presentable" if every probability σ in $\tilde{\mathbf{S}}$ is presentable. The object of the present paper is to find conditions under which $\tilde{\mathcal{X}}$ is presentable. Preliminaries: (Th. 3.2) Every $\sigma\in\tilde{\mathbf{S}}$ is invariant under all permutations of the co-ordinates. (Th. 5.3) In the linear space of all finite measures on $\tilde{\mathcal{X}}$, $\tilde{\mathbf{P}}$ is the set of extreme points of $\tilde{\mathbf{S}}$. (Th. 6.3) L is a normed linear space over the real numbers, L^* the conjugate space of L, N a weakly compact, convex subset of L. The set P of extreme points of N is assumed to be weakly closed. $\mathcal{M}$ is the family of weak Baire sets of P. Then every $n\in N$ can be represented in the form $\int_P g d\mu(g)$ where μ denotes a countably additive probability on $\mathcal{M}$. Main theorems: Infinite situation. (Th. 7.2) X is a compact Hausdorff space, $\mathcal{X}$ the Boolean σ-algebra of Baire sets in X. Then $\tilde{\mathcal{X}}$ is presentable. (Th. 7.4) Y is a locally compact Hausdorff space, X a wide Baire set (i.e., Baire set or complement) of Y. $\mathcal{X}$ consists of all wide Baire sets of Y that are contained in X. Then $\tilde{\mathcal{X}}$ is presentable. Finite

situation. (Th. 8.1) Every Boolean algebra $\mathfrak{X}$ of subsets of a set X is presentable. The proofs depend on the possibility of interpreting the linear space of all countably additive finite measures on $\mathfrak{X}$ or of the finite measures on $\mathfrak{X}$ of bounded variation as conjugate of a space L of functions on $\tilde{X}$ in order to apply Th. 6.3 with $N=\tilde{\mathbf{S}}$ and $P=\tilde{\mathbf{P}}$. For Th. 7.2 $L=\mathfrak{C}(\tilde{X})$, for Th. 8.1., L=space of all bounded $\tilde{\mathfrak{X}}$-measurable functions on X. The 1-point compactification is used to establish Th. 7.4. For technical simplicity § 9 and § 10 are confined to the infinite situation. In § 9 an algorithm based on the strong law of large numbers is given for computing μ corresponding to a given presentable σ; μ is unique. In § 10 the same algorithm applied to any σ leads to a new formulation of the condition of presentability. In § 11 $\tilde{X}$ denotes a set, $\tilde{\mathfrak{X}}$ a Boolean σ-algebra of subsets of $\tilde{X}$, G a class of isomorphisms of $\tilde{\mathfrak{X}}$ into itself, $\tilde{\mathbf{S}}$ the set of all probabilities on $\tilde{\mathfrak{X}}$ which are invariant under all transformations of G, $\tilde{\mathbf{P}}$ the set of extreme points of $\tilde{\mathbf{S}}$, $\mathcal{P}^*$ the Boolean σ-algebra of subsets of $\mathbf{P}$ generated by the sets $\{\tilde{\pi}; \tilde{\pi} \in \tilde{\mathbf{P}}, \tilde{\pi}A \leqq \lambda\}$ for λ real and $A \in \tilde{\mathfrak{X}}$. Th. 11.7. If G is a finite group, for every $\sigma \in \tilde{\mathbf{S}}$ there is exactly one countably additive measure μ defined on $\mathcal{P}^*$ such that $\sigma A=\int_{\tilde{\mathbf{P}}} (\tilde{\pi}A)d\tilde{\mu}(\tilde{\pi})$ for all $A \in \tilde{\mathfrak{X}}$. This μ is determined by the condition that $\mu\{\tilde{\pi}; \tilde{\pi} \in \tilde{\mathbf{P}}, \tilde{\pi}A=1\}=\sigma A$ for all $A \in \tilde{\mathfrak{X}}$ that are invariant under G. There is an interesting digression concerning Th. 11.3.: A product measure on an infinite product of measure spaces can assume only the values 0 and 1 for sets that are invariant under all finite permutation of the co-ordinates. In § 12 the authors study the presentability of finite Cartesian products in the infinite situation. *C. Pauc.*

Hájek, J.; and Rényi, A. Generalization of an inequality of Kolmogorov. Acta Math. Acad. Sci. Hungar. **6** (1955), 281–283. (Russian summary)

The authors prove the following theorem. If $\xi_1, \xi_2, \cdots, \xi_k, \cdots$ is a sequence of mutually independent random variables with mean values $M(\xi_k)=0$ and finite variances $M(\xi_k^2)=D_k^2$ $(k=1, 2, \cdots)$ and c_k $(k=1, 2, \cdots)$ is a non-increasing sequence of positive numbers, we have for any $\varepsilon>0$ and any positive integers n and m $(n<m)$

$$(2)\quad \mathrm{P}\Big(\max_{n\leq k\leq m} c_k|\xi_1+\xi_2+\cdots+\xi_k|\geqq\varepsilon\Big) \leqq \frac{1}{\varepsilon^2}\Big(c_n^2\sum_{k=1}^{n} D_k^2+\sum_{k=n+1}^{m} c_k^2D_k^2\Big).$$

J. Wolfowitz (Ithaca, N.Y.).

Thomasian, A. J. Distances et normes sur les espaces de variables aléatoires. C. R. Acad. Sci. Paris **242** (1956), 447–448.

Let $\mathfrak{X}$ be the space of real random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and f be a real-valued function defined on $\mathfrak{X}$. A sequence $\{X_n\}$ of elements of $\mathfrak{X}$ is said to "converge in f" to zero if $f(X_n)$ converges to zero. The following two theorems are announced. Theorem 1. The following four conditions concerning $(\Omega, \mathcal{A}, P)$ are equivalent: 1) There exists a function f such that "convergence in f" is equivalent to almost-everywhere convergence. 2) Every sequence of elements of $\mathfrak{X}$ which converges in probability converges almost everywhere. 3) There exists a function f satisfying 1) and $d(X, Y)=f(X-Y)$ is a metric for $\mathfrak{X}$. 4) Ω is the union of a countably many atoms. Theorem 2. The following three conditions concerning $(\Omega, \mathcal{A}, P)$ are equivalent: 1) There exists a function f such that "convergence in f" is equivalent to almost-everywhere convergence and such that $f(c1_A)=|c|f(1_A)$ (1_A is the characteristic function of A) for every event A. 2) There exists a norm f such that "convergence in f" is equivalent to almost everywhere convergence. 3) Ω is the union of finitely many atoms. *S. C. Moy.*

Dugué, Daniel. Incompatibilité de la convergence presque certaine et de l'écart. C. R. Acad. Sci. Paris **242** (1956), 728–729.

The following is proved. In the space $\mathfrak{X}$ of real random variables on a probability space if convergence almost everywhere and convergence in probability are distinct there exists no real function $\varphi(X, Y)$, $X \in \mathfrak{X}$, $Y \in \mathfrak{X}$ such that X_n converges almost everywhere to X if and only if $\varphi(X, X_n)$ converges to zero. *S. C. Moy.*

Franckx, Ed. La loi faible des grands nombres des variables stochastiques uniformément bornées. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 47–50.

Given random variables $(x_n: n>0)$ with common bound $B<+\infty$, set $s_n=(1/n)(x_1+\cdots+x_n)$, σ_n=variance (s_n), c_n=covariance (x_n, s_n). The author observes that $\sigma_n^2 \leqq (2/n^2)(c_1+2c_2+3c_3+\cdots+nc_n)$ and concludes that for the weak law of large numbers to hold it is necessary and sufficient that lim sup c_n be $\leqq 0$. Various minor mistakes occur. *H. P. McKean, Jr.* (Princeton, N.J.).

Rvačeva, E. L. On domains of attraction of multidimensional distributions. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. 6 (1954), 5–44. (Russian)

The author considers infinitely divisible and stable laws in s dimensions, extending many known results in the book by Gnedenko and Kolmogorov [Limit distributions for sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; MR **12**, 839; **16**, 52]. These extensions are rather straightforward though, and the author mentions as an open problem the study of distribution functions F such that

$$F(AX+B)=F(A_1X+B_1)*F(A_2X+B_2),$$

where A, A_1, A_2 are matrices of rank s, in the obvious context. *K. L. Chung* (Syracuse, N.Y.).

Meĭzler, D. G. On partial limit distributions for the maximal term of a variational series. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. 1 (1955), 24–44. (Russian)

Let $\{\xi_n, n\geqq 1\}$ be a sequence of independent random variables with the common distribution F. Let $\eta_n=\max_{1\leq k\leq n}\xi_k$. If there exists an increasing sequence of integers n_k, and constants $a_k>0$ and b_k such that $\lim_{k\to\infty} F^{n_k}(a_kx+b_k)=\Phi(x)$, where Φ is a distribution function (d.f.), then Φ is said to have a domain of partial attraction to which F belongs. The terminology and notation are analogous to those for the sums of ξ_n. If $\lim_{k\to\infty} n_k/n_{k+1}=r$, then r is the rank of the partial attraction. The following results are proved. (1) Every d.f. has a (non-empty) domain of partial attraction (p.a.). (2) Φ has a domain of p.a. of rank r if and only if there exist α, β such that $\Phi^{1/r}(\alpha x+\beta)=\Phi(x)$. (3) For $r=1$ only the three types of d.f.'s given by Gnedenko [Ann. of Math. (2) **44** (1943), 423–453; MR **5**, 41] have a domain of p.a. (4) Φ is one of these three types if and only if it has two ranks of p.a. whose logarithms are incommensurable; if Φ

has a domain of p.a. of positive rank but does not belong to one of the three types then all ranks of its p.a. are of the form $r=r_0^m$ $(m=1, 2, \cdots)$, where r_0 $(0<r_0<1)$ is the maximum rank. (5) If F belongs to the domains of p.a. of Φ of rank r $(0<r<r_0)$, then it also belongs to the maximum rank r_0. There are two more theorems concerning special types, one of which gives a necessary and sufficient condition that a differentiable d.f. has a domain of p.a. of positive rank. There are several examples, one of which is a d.f. which does not belong to the domain of p.a. of any d.f. *K. L. Chung* (Syracuse, N.Y.).

Chang, Li-Chien. On the ratio of an empirical distribution function to the theoretical distribution function. Acta Math. Sinica **5** (1955), 347–368. (Chinese. English summary)

Let $F(x)$ be a distribution function and $F_n(x)$ the corresponding empiric distribution function based on n samples. It is shown that

$$\Pr\left\{\sup\left|\frac{F_n(x)}{F(x)}-1\right|\geqq\varepsilon\right\}\to 0 \text{ as } n\to\infty,$$

where the sup is over the set of x for which $F(x)$ (or $F_n(x)$) $\geqq a_n/n$ and a_n is any sequence of positive numbers $\to\infty$. Exact formulas are obtained for

$$\Pr\left\{\sup_{0<F(x)\leqq a}\frac{F_n(x)}{F(x)}<z\right\} \text{ and } \Pr\left\{\inf_{0<F_n(x)\leqq a}\frac{F_n(x)}{F(x)}<z\right\}$$

under the assumption that either (i) $a=1$ and F is continuous, or (ii) $a<1$, $F(x)=a$ has a solution and F is continuous on the set $\{F(x)\leqq a\}$. Limits of the above probabilities are obtained if $a=c/n$ or a_n/n, where c is a positive constant and a_n is as before; under the assumption that there exists p, $0<p<1$, such that $F(x)=p$ has a solution and F is continuous on $\{F(x)\leqq p\}$. These limits do not depend on p or a_n. *K. L. Chung.*

Dynkin, E. B. Some limit theorems for sums of independent random quantities with infinite mathematical expectations. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 247–266. (Russian)

Let $\{\xi_n, n\geqq 1\}$ be a sequence of independent, identically distributed positive random variables and $\zeta_n=\sum_{v=1}^n \xi_v$. Let v_x be the number of n such that $\zeta_n<x$; $\gamma_x'=\zeta_{v_x+1}-x$, $\gamma_x''=x-\zeta_{v_x}$, $\gamma_x=\xi_{v_x+1}=\gamma_x'+\gamma_x''$. If $M\xi_k=\mu$ is finite and the distribution (of ξ_1) is non-lattice, then

$$\lim_{x\to\infty} P(\gamma_x'>u)=\int_u^\infty (1-F(z)\mu^{-1}dz.$$

Other relations follow from $P(\gamma_x''>u)=P(\gamma_{x-u}'>u)$, $P(\gamma_x'>u, \gamma_x''>v)=P(\gamma_{x-v}'>u+v)$. Now suppose $M\xi_k=\infty$. Let $c(x)$ be such that for every $k>0$, $\lim_{x\to\infty} c(kx)/c(x)$ exists and $\neq 0$. If $\lim_{x\to\infty} c(x)/x$ is 0 or ∞, then $\gamma_x'/c(x)$ cannot have any proper limit distribution. If $\lim_{x\to\infty} c(x)/x=1$, then all proper limit distributions are from the family with density $p_\alpha(u)=(\sin\pi\alpha)\pi^{-1}u^{-\alpha}(1+u)^{-1}$ if $u>0$ $(0<\alpha<1)$, $=0$ if $u\leqq 0$. Limit joint distribution for γ_x' and γ_x'', and that for γ_x'' and for γ_x are also given. Extensions to processes with independent increments and to not necessarily positive ξ_n are discussed. The results are obtained by the method Laplace transform.

K. L. Chung (Syracuse, N.Y.).

Hammersley, J. M. The area enclosed by Pólya's walk. Proc. Cambridge Philos. Soc. **52** (1956), 78–87.

The author considers the ordinary random walk of unit step in the plane and denotes by T_n the track marked out in the first n steps. He defines the interior of T_n to be the set of points which neither lie on T_n nor can be connected to the point at infinity without crossing T_n, and denotes by A_n the area of this set. A_n is a non-negative random variable and the author's main result states that for almost all paths we have

$$(*)\qquad \left(\frac{\pi}{128}-\varepsilon\right)n/\log n<A_n<(4+\varepsilon)n\log\log n$$

as $n\to\infty$ for every $\varepsilon>0$. The right-hand inequality follows directly from the law of the iterated logarithm while the proof of the other inequality is more involved and uses, among other things, estimates developed by Dvoretzky and Erdös [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 353–367; MR **13**, 852]. The paper closes with a few conjectures and a numerical computation. In a footnote it is asserted that P. Lévy succeeded in replacing the constant 4 in (*) by $1/4\pi$ and showing that no smaller constant would do. [The following misprints were noticed by the reviewer: A factor 2 is missing before the Σ in (3) on p. 79; c_2 should be replaced by c_1 in the last displayed formula on p. 83.] *A. Dvoretzky* (Jerusalem).

Ramakrishnan, Alladi. Processes represented as integrals of a class of random functions. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 120–127.

A la fonction aléatoire $x(\tau)$, on associe la fonction

$$y_m(t)=\int_0^t \phi_m(\tau_m)d\tau_m\cdots\int_0^{\tau_3}\phi_2(\tau_2)d\tau_2\int_0^{\tau_2}\phi_1(\tau_1)x(\tau_1)d\tau_1 = \int_0^t F(t,\tau)x(\tau)d\tau;$$

les ϕ_t sont des fonctions données, d'où se déduit F. On se propose de calculer la densité de probabilité ou les moments de $y_m(t)$, par une méthode plus simple que ne le suggérait un travail antérieur [mêmes Proc. **58** (1955), 470–482, 634–645; MR **17**, 637].

On y arrive en utilisant comme intermédiaire

$$y_m(t,a)=\int_0^t \dot{F}(a,\tau)x(\tau)d\tau.$$

Si $x(t)$ est un processus de Markoff, la densité de probabilité conjuguée de $y_m(t,a)$ et $x(t)$ vérifie une équation fonctionnelle assez simple, dépendant de la probabilité de transition $R(x'|x)dx'dt$ de $x(t)$ dans l'intervalle dt. Des expressions sont données pour les moments d'ordres 1, 2, 3 de $y_m(t)$, lorsque $x(t)$ est un „processus ponctuel" non poissonnien. Dans le cas poissonnien, on obtient aussi, par sa transformée de Laplace, la densité de probabilité de $y_m(t)$.

Ces résultats permettent d'étudier des équations différentielles linéaires à coefficients constants dont le second membre est une fonction aléatoire donnée. *J. Bass.*

Mathews, P. M.; and Srinivasan, S. K. Ordinary linear differential equations involving random functions. Proc. Indian Acad. Sci. Sect. A. **43** (1956), 4–20.

Les auteurs considèrent l'équation différentielle

$$(1)\qquad \frac{d^m y}{dt^m}+\lambda_1(t)\frac{d^{m-1}y}{dt^{m-1}}+\cdots+\lambda_m(t)y=k(t)x(t),$$

où $x(t)$ est une fonction aléatoire donnée. Ils supposent qu'on a pu mettre la solution sous la forme

$$(2)\ y=\phi_m(t)\int_0^t \phi_{m-1}(t_{m-1})dt_{m-1}\cdots\int_0^{t_1}\phi_0(t_0)k(t_0)x(t_0)dt_0.$$

L'étude des propriétés stochastiques de y est alors possible, tout au moins si $x(t)$ a une structure assez simple.

Le cas d'une équation (1) du premier ordre est d'abord examiné, avec divers exemples de seconds membres, correspondant à des phénomènes physiques intéressants. Le cas d'une équation (1) du second ordre est ensuite traité d'une façon analogue. On montre en annexe qu'il est possible de mettre la solution sous la forme (2) et qu'il est d'ailleurs commode de l'écrire sous la forme d'une opération intégrale linéaire simple

$$y=\int_0^t k(\tau)f(t,\tau)x(\tau)d\tau$$

(le rapporteur signale une erreur à l'appendice A. Le second membre de la formule A-2 doit être multiplié par $e^{\lambda\rho(t)}$, ce qui modifie la formule 34). Le cas général d'une équation d'ordre n est enfin abordé. *J. Bass* (Paris).

Takács, L. Some investigations concerning recurrent stochastic processes of a certain type. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 115–128 (1955). (Hungarian. Russian and English summaries)

This is an expansion of some of the author's earlier investigations [same Közl. 2 (1953), 135–151; Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4 (1954), 525–541; MR **16**, 379, 723]. A system is considered which can be in any one of the states $\{E_j\}$. The number of possible states is finite or denumerable; transitions from one state to another have certain transition probabilities. Transitions occur randomly and the following assumptions are made concerning the time points t_n at which transitions occur: (i) A transition occurs at $t=0$; (ii) the time intervals t_n-t_{n-1} between transitions are independent positive random variables; the distribution of t_n-t_{n-1} depends only on the state assumed by the system at the time t_{n-1}. The author studies the distribution of the number of transitions $E_j\to E_k$ during a time interval of length t and the distribution of the time spent in state E_k. *E. Lukacs.*

Dynkin, E. B. On some limit theorems for Markov chains. Ukrain. Mat. Ž. 6 (1954), 21–27. (Russian)

Let $\{x(t), t\geqq 0\}$ be a Markov process. The author discusses the asymptotic distribution of random variables of the form $\zeta(u)=\sum_0^u f[x(t),t]$ for large u, and of the corresponding integral in the continuous parameter case. The following new results are announced in one case discussed. Let $x(t)$ be the position at time t of a particle in a random walk (proceeding in independent unit steps with probabilities $\frac{1}{2}$, $\frac{1}{2}$ for steps ± 1) and suppose that $\sum_{-\infty}^{\infty} f(k)=\infty$. Then, if $f(k,t)=f(k)$ above, and if $\zeta(u)/B(u)-A(u)$ has an asymptotic distribution, for some sequences A, B, with $\lim_{u\to\infty} B(u)=\infty$, the same is true, with the same A, B and limit distribution, if f is replaced by g, where $\lim_{|k|\to\infty} g(k)/f(k)=1$. This result leads to the theorem that, if $\alpha>-1$, and if $\lim f(k)|k|^{-\alpha}$ is $c_1\neq 0$ for $k\to+\infty$, and is c_2 for $k\to-\infty$, then $\zeta(u)/u^{1+1\alpha}$ has a limit distribution. This distribution is the distribution of $\int_0^1\Phi(X(t))dt$, where the $X(t)$ process is the Wiener (Brownian motion) process, and $\Phi(x)=c_1x^\alpha$ or $c_2|x|^\alpha$ according as $x>0$ or $x<0$. *J. L. Doob* (Urbana, Ill.).

Dynkin, E. B. Continuous one-dimensional Markov processes. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 405–508. (Russian)

The author generalizes the results of the paper reviewed above. He assumes again that he has a strict Markov process with state space a closed linear interval, and with continuous sample functions. He supposes that T_t takes B into B, but, generalizing his previous paper, and the work of Feller already referred to, he no longer supposes that T_t takes C into C. Let H be the subspace of B whose functions have only discontinuities of the first kind, are continuous on the left (right) at every point which is accessible from the left (right) and even continuous on both sides at a point which is accessible from one side only unless a certain further accessibility type condition is satisfied. Here a point x is called accessible from the left (right) if there is a point $a_1<x$ ($a_2>x$) for which the probability of a trajectory ever reaching x or beyond is positive. The points of the state interval are divided into several classes. For example, one such class consists of the points which are transitional in both directions, but accessible from neither direction. For x in each class, an operation $(\mathfrak{A}f)(x)$ is defined, an explicit derivation operation. It is stated that H is a closed subspace of B, invariant under T_t, that (using H from now on as the domain of the T_t semigroup) the semigroup is strongly continuous at 0, and that the infinitesimal operator of the semigroup is a contraction of $\mathfrak{A}$. Moreover A is written in the form D_vD_u, where the derivations are with respect to specified functions, u v. *J. L. Doob.*

Dynkin, E. B. Infinitesimal operators of Markov random processes. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 206–209. (Russian)

Let $\{x(t), 0\leqq t<\infty\}$ be a Markov process with stationary transition probabilities and a metric state space. Let M_x signify conditional expectations for $x(0)=x$, and define $(T_tf)(x)=M_xf[x(t)]$, for f in the Banach space B of bounded Borel measurable functions on the state space. It is supposed that T_t takes B into B. Then the T_t family is a semigroup. Let A be the corresponding infinitesimal operator. It is supposed that almost all sample functions of the stochastic process have limits from each direction at all parameter points, and are right continuous. For each open set U, let τ_U be the first parameter value for which a sample function does not have its value in U. It is supposed (strict Markov property) that, for each U which is a neighborhood of a point x, the process $\{x(\tau_U+t), t\geqq 0\}$ is a Markov process with the same transition probabilities as the $x(t)$ process, and is independent of the original process from x, stopped at τ_U, relative to $x(\tau_U)$. It is proved that, if f is in the domain of A, if Af is continuous at x, and if, for some neighborhood U^0 of x, $M_x\tau_{U^0}<\infty$, then

$$(Af)(x)=\lim\frac{M_xf[x(\tau_U)]-f(x)}{M_x\tau_U},$$

where the limit is taken as the diameter of U goes to 0. Now suppose further that the semigroup takes the space C of bounded continuous functions on the state space into itself, and is strongly continuous at the parameter value 0, and, from now on, consider the semigroup and A on C only. It is proved that then, if the state space is compact, f is in the domain of A if and only if the above limit exists on the state space, and defines a continuous function.

In particular, suppose that the state space is a closed linear interval, and that almost all sample functions are continuous. A point x is called transitional to the right [left] if, starting from x there is positive probability of entering the half-line $(x,\infty]$ $[[-\infty,x)]$ at some time t. If x is neither, Af vanishes at x whenever Af is defined.

Excluding this case, there is a neighborhood U^0 of x such that, if $y \in U^0$, then $m(y)=M_{y\tau U^0}<\infty$. If x is transitional to the right but not to the left, $(Af)(x)=-d^+f(x)/dm$. If f is transitional to both sides, and if $p(y)$ is the probability of reaching a_2 before a_1, starting from y, then it is stated that it is possible to choose $a_1<x<a_2$ in such a way that $p(y)>0$ in (a_1, a_2) and that, with this choice, $(Af)(x)=D_nD_pf(x)$, where D_p means differentiation with respect to p and D_n with respect to $n=dm/dp$. These formulas were obtained in a slightly different context, without the explicit probability interpretation, by Feller [Ann. of Math. (2) **61** (1955), 90–105; MR **16**, 824]. *J. L. Doob.*

van Dantzig, D. Chaînes de Markof dans les ensembles abstraits et applications aux processus avec régions absorbantes et au problème des boucles. Ann. Inst. H. Poincaré **14** (1955), 145–199.

In connection with the study of the probability of non-appearance of a certain "catastrophe" of a stationary Markov chain with an absorbing barrier A an operator on the collection of bounded measurable functions on the state space E is introduced. Applications to the study of the problem of loops are given. In an appendix, the theory of generalized matrices is developed. (If E and E' are two sets and δ_E and $\delta_{E'}$ are δ-fields of subsets of E and E' respectively, a generalized matrix P_X^x is a real- or complex-valued function on $E\times\delta_{E'}$ such that, for each fixed $X \in \delta_{E'}$, P_X^x is a bounded δ_E-measurable function on E and, for each fixed $x \in E$, P_X^x is a countably additive set function on $\delta_{E'}$.) As a generalization of a process with stationary independent increments, a Markov transition-probability matrix invariant under a group of transformations on the state space E is studied. A representation theorem is given. A generalization of A. Wald's fundamental equality is also given. *S. C. Moy.*

Maravall Casesnoves, Dario. Impulsive random motion and hereditary stochastic processes. Rev. Mat. Hisp.-Amer. (4) **15** (1955), 9–30. (Spanish)

In this elementary paper some simple examples of Markoff processes are presented, together with a few remarks on non-markoffian processes.

N. G. van Kampen (Utrecht).

Urbanik, K. Some remarks on the asymptotic behaviour of the cosmic ray cascade for large depth of the absorber. I. Estimation of the factorial moments. Nuovo Cimento (10) **2** (1955), supplemento, 1147–1149.

Consider a cascade of nuclear particles or "nucleons." A nucleon has probability dx of producing a pair of new nucleons in the thickness dx; the two secondaries have fractions ε' and ε'' of the original energy, with probability density $w(\varepsilon', \varepsilon'')$; it is assumed that $w=0$ if $\varepsilon'+\varepsilon''>1$ or $\varepsilon'<\varepsilon''$. Let $P(n|\varepsilon, x)$ be the probability that there are exactly n nucleons of energy $>\varepsilon$ at thickness x, starting with one particle of unit energy at thickness 0. Let $S_k(\varepsilon, x)$ be the kth factorial moment,

$$S_k=\sum_{n=k}^{[\varepsilon^{-1}]} n(n-1)\cdots(n-k+1)P(n|\varepsilon, x).$$

It is shown that $S_k(\varepsilon, x)=\varepsilon^{-x}W_k(\varepsilon, x)$ [the text misprints ε^{-k} for ε^{-x}], $k=1, 2, \cdots$, where $\lim_{x\to\infty} W_k(\varepsilon, x)=\infty$ but for each $\beta>0$, $\lim_{x\to\infty} e^{-\beta x}W_k(\varepsilon, x)=0$. Use is made of an integral equation satisfied by S_1. *T. E. Harris.*

Łopuszański, J. Some remarks on the asymptotic behaviour of the cosmic ray cascade for large depth of the absorber. II. Asymptotic behaviour of the probability distribution function. Nuovo Cimento (10) **2** (1955), supplemento, 1150–1160.

Notation as in the previous review. For any $\varepsilon>0$, $S_1(\varepsilon, x)\to 0$ as $x\to\infty$; hence with probability 1 the cascade dies out. Let $\Gamma(\varepsilon, 0, x)=\sum_{n=1}^{\infty} P(n|\varepsilon, x)$ and let $P^*(n|\varepsilon,x)=P(n|\varepsilon, x)/\Gamma(\varepsilon, 0, x)$, $n=1, 2, \cdots$. It is shown that $\lim_{x\to\infty} P^*(1|\varepsilon, x)=1$; also if $\varepsilon'>\varepsilon>0$, then

$$\Gamma(\varepsilon', 0, x)/\Gamma(\varepsilon, 0, x)\to 0.$$

The meaning is that if a particle of energy $>\varepsilon$ exists at a great thickness x, then with high probability there is only one such particle and its energy is close to ε. [The result may be compared with one of Yaglom for simple branching processes with probability 1 of extinction, Dokl. Akad. Nauk SSSR (N.S.) **56** (1947), 795–798; MR **9**, 149. However, in Yaglom's case, the limiting conditional distribution is nondegenerate.] The proof makes use of an integro-differential equation satisfied by the generating function $G(\varepsilon, u, x)=\sum_{n=0}^{\infty} P(n|\varepsilon, x)u^n$, namely:

$$\partial G(\varepsilon, u, x)/\partial x+G(\varepsilon, u, x)=\int_0^1\int_0^1 w(\varepsilon', \varepsilon'')G(\varepsilon/\varepsilon', u, x) \times G(\varepsilon/\varepsilon'', u, x)d\varepsilon'd\varepsilon'',$$

attributed to L. Jánossy [Proc. Phys. Soc. Sect. A. **63** (1950), 241–249]. *T. E. Harris.*

Łopuszański, J. Some remarks on the asymptotic behaviour of the cosmic ray cascade for large depth of the absorber. III. Evaluation of the distribution function. Nuovo Cimento (10) **2** (1955), supplemento, 1161–1167.

Notation as in the two preceding reviews. It is shown that if $\varepsilon>0$ and $n>m\geqq 1$, then $\lim_{x\to\infty} S_n(\varepsilon, x)/S_m(\varepsilon, x)=0$, and $P(n|\varepsilon, x)\sim S_n(\varepsilon, x)/n!$, $x\to\infty$. The importance of this is that in many cases the factorial moments can be estimated directly more readily than the probabilities. The author states that the two-particle cosmic ray cascade can be treated in a similar way. *T. E. Harris.*

Takács, L. On stochastic processes connected with certain physical recording apparatuses. Acta Math. Acad. Sci. Hungar. **6** (1955), 363–380. (Russian summary)

English version of Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4** (1954), 571–587 [MR **16**, 723].

Husu, A. P. On some functionals given on processes which are encountered in engineering. Vestnik Leningrad. Univ. **11** (1956), no. 1, 89–100. (Russian)

The paper is, for the most part, devoted to an expository heuristic discussion of the statistics of some common functionals of stationary Gaussian processes. In § 6 an approximate formula for the mean vertical distance between successive maxima and minima is given.

E. Reich (Santa Monica, Calif.).

Downton, F. Waiting time in bulk service queues. J. Roy. Statist. Soc. Ser. B. **17** (1955), 256–261.

Bulk service here is as defined by Bailey [same J. **16** (1954), 80–87; MR **16**, 148]; customers arriving at random before a single server are served in order of arrival in batches of at most s, the batch being less than s only when the queue is less than s when service is begun. Bailey's work is continued by expressing the Laplace transform of the equilibrium waiting time distribution in terms of the generating function for the probability of queue length.

This is examined in detail for Erlangian (χ^2) distributions of the service time, including the limiting case of constant service time. Two illustrative numerical tables are given, one for the mean waiting time, the other for its variance. *J. Riordan* (New York, N.Y.).

Longuet-Higgins, M. S. Statistical properties of a moving wave-form. Proc. Cambridge Philos. Soc. **52** (1956), 234–245.

The author considers a moving one-dimensional wave specified by a stochastic function $f(x, t)$, which he takes to be a Gaussian random process. With the motivation of explaining optical phenomena commonly observed with water waves, he examines the dependence of certain statistics of f on the energy spectrum. Among the quantities considered are the rate of motion of the zeroes of f, twinkles (creation or annihalation of a pair of specular points), winkles (zeroes of order two), and crinkles (creation of a pair of inflection points). The case when the spectrum is a narrow band is considered in detail. *E. Reich* (Santa Monica, Calif.).

de Finetti, Bruno. La compensazione tra rischi eterogenei. Giorn. Ist. Ital. Attuari **17** (1954), 1–21.

"Opinions about the advantage or disadvantage of having to deal with homogeneous as opposed to heterogeneous risks are examined from the conceptual point of view but with reference to actuarial practice. It is concluded that homogeneity is useful in the gathering of statistics but indifferent, or harmful, so far as compensation among random departures from expectation, and therefore so far as stability, is concerned. With reference to this situation some considerations about a few currently controversial points about the theory of risk are added." [Author's summary translated by reviewer.]

The discussion is penetrating though amathematical. One of the supplementary points is especially interesting. It is fashionable and mathematically intriguing to discuss reserve policies under which the probability is small that an insurance company will be ruined in any time however large. [See e.g., H. Cramér, Univ. California Publ. Statist., **2** (1954), 99–123; MR **16**, 494.] The author is a pioneer in such studies, but he here advances a strong argument that they are an altogether unrealistic approach to the practical problem of risk, and he suggests a simple alternative approach. *L. J. Savage* (Chicago, Ill.).

Good, I. J. Some terminology and notation in information theory. Proc. Inst. Elec. Engrs. C. **103** (1956), 200–204.

The main purpose of the paper is to stimulate thought concerning terminology, notation and exposition of some basic parts of information theory. The notation used here is intended to be simple, nearly self-explanatory, readable in words from left to right, and suggestive of new applications. Moreover, sufficient generality is preserved to ensure that entropy can be interpreted without necessarily depending on the frequentist definition of probability (as a limiting frequency in an infinite sequence of trials). Mention is made of some connections of the theory with inverse probability and with mathematical statistics in general. [Author's summary.]

The main idea is to begin with the information in a proposition H defined as $-\log P(H)$, so that the usual information concept becomes "expected information." The "amount of information concerning H provided by E" is defined as $I(H:E)=-\log P(H)-[-\log P(H|E)]=\log\{P(E \text{ and } H)/P(E)P(H)\}$.

The properties of I are illustrated by the fact that (under discreteness assumptions) if $\hat{\theta}(X)$ is sufficient for θ then $I(\theta:X)=I(\theta:\hat{\theta})$. "This connection between sufficient statistics and information is simpler than one that has been suggested previously, using expected amounts of information." *L. J. Savage* (Chicago, Ill.).

See also: Baatard, p. 818; del Busto, p. 818; von Juhos, p. 818; Kamalov, p. 871.

Mathematical Statistics

Fisher, Ronald. Statistical methods and scientific induction. J. Roy. Statist. Soc. Ser. B. **17** (1955), 69–78.

The author restates and clarifies his position on the principles underlying statistical scientific inference, contrasting its problems with the technological problems typified by acceptance sampling. Many views which could formerly be attributed to him by inference are now set forth explicitly, and the basic issues are more clearly drawn. *J. W. Tukey* (Princeton, N.J.).

Pearson, E. S. Statistical concepts in their relation to reality. J. Roy. Statist. Soc. Ser. B. **17** (1955), 204–207.

The author responds to R. A. Fisher's discussion [see the paper reviewed above] of statistical scientific inference with a clear and illuminating restatement of the origins, attitudes, and philosophy of the Neyman-Pearson theory. *J. W. Tukey* (Princeton, N.J.).

Breny, H. L'état actuel du problème de Behrens-Fisher. Trabajos Estadist. **6** (1955), 111–131. (Spanish summary)

This paper contains 1) the author's interpretation of the ideas of fiducial inference, 2) an outline of Fisher's solution to the "Behren's Fisher" (BF) problem, 3) a critical survey of the historical controversy surrounding fiducial inference and the BF problem, 4) a survey of "non-fiducial" exact and approximate solutions to the BF problem. *M. Dwass* (Evanston, Ill.).

Tukey, John W. Keeping moment-like sampling computations simple. Ann. Math. Statist. **27** (1956), 37–54.

The author develops a systematic notation for certain symmetric polynomials, and derives formulas to make it convenient to use these polynomials in commonly occurring calculations. The many details cannot be summarized in any compact manner. *L. Weiss.*

Hooke, Robert. Symmetric functions of a two-way array. Ann. Math. Statist. **27** (1956), 55–79.

The author extends Tukey's development [see the paper reviewed above] to symmetric polynomials in the elements of a matrix. *L. Weiss* (Charlottesville, Va.).

Hooke, Robert. Some applications of bipolykays to the estimation of variance components and their moments. Ann. Math. Statist. **27** (1956), 80–98.

The author applies results of the paper reviewed above to develop unbiased estimators for variance components and for the variances of these estimators, in certain analysis-of-variance problems. *L. Weiss.*

Keeping, E. S. Statistical decisions. Amer. Math. Monthly **63** (1956), 147–159.

Expository paper.

Moore, P. G. The mean successive difference in samples from an exponential population. Trabajos Estadist. 6 (1955), 133–141. (Spanish summary)

Bystrov, N. F. On some unbiased estimates. Vestnik Leningrad. Univ. 11 (1956), no. 1, 169–175. (Russian) Expository paper. *J. Wolfowitz* (Ithaca, N.Y.).

Mokashi, V. K. Efficiency of stratification in sub-sampling designs for the ratio method of estimation. J. Indian Soc. Agric. Statist. 6 (1954), 77–82.

For the evaluation of this efficiency the author constructs an estimate of the difference between the variances of two mean estimates based one on a non-stratified sample and the other on a stratified sample.

D. M. Sandelius (Göteborg).

Singh, Daroga. On efficiency of the sampling with varying probabilities without replacement. J. Indian Soc. Agric. Statist. 6 (1954), 48–57.

From a population divided into N clusters of units n clusters are sampled without or with replacement in the following way. Let $p_1, \cdots, p_N$ denote positive numbers associated with the clusters. The probability of obtaining in the $(1+i)$st draw a particular cluster from the set of clusters remaining after the ith draw should be equal to

(the p-value for this cluster)/(the sum of the p-values for the set),

($i=0, 1, \cdots, n-1$). From each cluster sampled a simple random sample of units should be drawn without replacement. Extending a result by Narain [same J. 3 (1951), 169–174; MR 13, 570] the author determines an approximate criterion for comparing the replacement and non-replacement cases of cluster sampling with respect to variances of corresponding unbiased mean estimates. According to this criterion, sampling without replacement is preferable when $n=2$, while this is not necessarily true for $n>2$. *D. M. Sandelius* (Göteborg).

Yamamoto, Sumiyasu. On the theory of sampling with probabilities proportionate to given values. Ann. Inst. Statist. Math., Tokyo 7 (1955), 25–38.

Let π_i be the ith stratum of a population π, A_{ij} the jth cluster of π_i and N_{ij} the size or number of individuals of A_{ij} ($i=1, \cdots, R$; $j=1, \cdots, M_i$). Let $B_{i\alpha}$ be the αth of the possible combinations of m_i of the clusters of π_i. The author considers a sampling scheme which for π_i consists in choosing one of the $B_{i\alpha}$ and then n_i individuals from each A_{ij} included in the sample. In the first stage any particular $B_{i\alpha}$ will have the probability $p_{i\alpha}$ of being sampled, while in the second stage all individuals will have equal probabilities of being sampled. For given $p_{i\alpha}$, m_i and n_i the author derives that consistent unbiased linear estimate of the mean θ of a numerical characteristics of the indivuals in π which has minimum variance. When (*) the $p_{i\alpha}$ are proportionate to the sizes of the $B_{i\alpha}$ this estimate is identical with the one obtained by Midzuno [same Ann. 3 (1952), 99–107; MR 14, 390]. The author compares the variance in this case with the variance in the case (**) when, for each i, all $p_{i\alpha}$ are equal, and gives practical reasons for preferring (*) to (**) when the N_{ij} vary within the strata. Finally the author determines optimum values of $p_{i\alpha}$ and finds conditions for (*) and (**) to be optimum. *D. M. Sandelius* (Göteborg).

Chernoff, Herman. Large-sample theory: parametric case. Ann. Math. Statist. 27 (1956), 1–22.

This paper is largely expository, and was presented as an invited address at the 1954 Annual meeting of the Institute of Mathematical Statistics. It deals with a number of quite separate problems from the large sample point of view. Section 2 describes a calculus of stochastic limit and order relationships. In section 3, some asymptotic expressions are given for $P(\bar{X}_n<c)$, where $\bar{X}_n$ is an average of n independent identically distributed random variables with mean u and $c<u$. These two sections provide many of the technical tools used throughout the paper. Section 4, Estimation, is a survey of results, both previously known and new, on the sense in which maximum-likelihood estimates are asymptotically "efficient." Sections 5, 6, 7 are entitled: 5, Optimal designs for estimating parameters; 6, Testing simple hypotheses; 7, Composite hypotheses, and they essentially summarize work already published by the author [same Ann. 24 (1953), 586–602; 23 (1952), 493–507; 25 (1954), 573–578; MR 15, 452, 241; 16, 381]. It is worth mentioning that the results on asymptotic efficiency of a test given in section 6 are improvements and extensions of the previously published version referred to. *M. Dwass.*

Rubin, Herman. Uniform convergence of random functions with applications to statistics. Ann. Math. Statist. 27 (1956), 200–203.

If $X_1, X_2, \cdots$ are independently and identically distributed, evident conditions imply that, with probability 1, $n^{-1}\sum_1^n f(t, X_i) \to Ef(t, X_1)$ uniformly in t. The author uses this fact to obtain a consistent sequence of estimators for the problem of linear structure, using characteristic functions (c.f.) in a manner similar to that of Neyman [same Ann. 22 (1951), 497–512; MR 13, 481] but using a different functional of the sample c.f. [the condition $\lambda(\varepsilon)-\lambda(-\varepsilon)>0$ on p. 201 is not quite sufficient]. [See also J. Wolfowitz, Ann. Inst. Stat. Math., Tokyo 5 (1953), 9–23; MR 15, 452.] It is also shown that a continuous function on a set of distribution functions (Levy topology) can be consistently estimated [also obtainable from the Glivenko-Cantelli theorem]. Similar results are obtained for continuously identified parameters. *J. Kiefer* (Ithaca, N.Y.).

Healy, M. J. R. A significance test for the difference in efficiency between two predictors. J. Roy. Statist. Soc. Ser. B. 17 (1955), 266–268.

Given variates $x_1, \cdots, x_p$, it is required to test whether any of them are more effective predictors than others of a dependent variate y. Effectiveness is judged by the smallness of the residual sum of squares. This test was originally derived by H. Hotelling [Ann. Math. Statist. 11 (1940), 271–283; MR 2, 111], using the correlation approach. The present paper gives an alternative derivation using the concepts of regression and analysis of variance and puts the result into a form that can be readily computed. *S. W. Nash* (Vancouver ,B.C.).

Jowett, G. H. The comparison of means of sets of observations from sections of independent stochastic series. J. Roy. Statist. Soc. Ser. B. 17 (1955), 208–227.

Methodology for performing tests of hypotheses about mean values of discrete time series is presented with several detailed examples. The main device used allows the statistician to judge from the results of appropriate computations when observations are sufficiently separated

in "time" to be considered uncorrelated. [For previous work using similar techniques see Jowett, Applied Statistics **4** (1955), 32–46; and Jowett and Scott, J. Roy. Statist. Soc. Ser. B. **15** (1953), 81–86]. *I. R. Savage.*

Kitagawa, Tosio. Errata: Some stochastic considerations upon empirical functions of various types. Bull. Math. Statist. **6** (1955), no. 1–2, 49–50.
See same Bull. **5** (1953), no. 3–4, 19–33 [MR **15**, 886].

Washio, Yasutoshi. On the weighted power function of some testing hypotheses. Bull. Math. Statist. **6** (1955), no. 1–2, 11–15.
A well-known characterization of the size α test of a hypothesis which is a Bayes solution in the sense of maximizing a weighted average of power with respect to an a priori distribution over the set of alternatives, is given for some special cases. There are examples involving the normal distribution. *M. Dwass* (Stanford, Calif.).

Takács, L. Anwendung wahrscheinlichkeitstheoretischer Methoden bei der Untersuchung gewisser meteoropathologischer Erscheinungen. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 301–320 (1955). (Hungarian. Russian and German summaries)
The author studies a test for the hypothesis that one sequence of events (e.g., biological phenomena) is not influenced by the occurrence of another sequence of events (e.g. meteorological phenomena). Denote the times of occurrence of the first sequence by $\{t_n\}$ and those of the second by $\{u_n\}$. Observations are taken over a time interval T which is divided into $2m+1$ subintervals, centered around the origin. H. v. Schelling suggested in a paper not available to this reviewer [Ergebnisse der Hygiene, Bakteriologie, Immunitätsforschung und Experimentelle Therapie **24** (1941), 87–149 the following procedure for the construction of a test statistic: For each t_n the points $u_k - t_n$ are marked on the time axis and the number falling into each of the $2m+1$ intervals is counted. Let v_j^* be the number of observations falling into the jth ($j=0, \pm 1, \cdots, \pm m$) interval. H. v. Schelling proposed to use $\max_i v_i^* - \min_i v_i^*$ as a test statistic and outlined, without proof, a procedure and gave tables for testing the hypothesis. The present author assumes that the two series are independent and studies the random variables v_j ($j=0, \pm 1, \cdots, \pm m$) which denote the number of points falling into the jth interval. The first- and second-order moments of the joint distribution of these random variables are determined under the following two alternative assumptions: (1) Each of the series $\{u_n\}$ as well as $\{t_n\}$ comes from a Poisson process; (2) the process of the $\{u_n\}$ is a Poisson process while the $t_j - t_{j-1}$ are independently and identically distributed, their distribution function is not a lattice distribution and its moments of the second order exist. To get an approximate distribution for the test statistic the author has to assume the normality of the joint distribution of the v_j; in his proof he uses the results concerning secondary processes generated by a primary process which he developed in an earlier paper [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 187–197; MR **17**, 276]. *E. Lukacs* (Washington, D.C.).

Good, I. J. On the weighted combination of significance tests. J. Roy. Statist. Soc. Ser. B. **17** (1955), 264–265.
The distribution of $Q = P_1^{\lambda_1} P_2^{\lambda_2} \cdots P_n^{\lambda_n}$ is obtained, where $P_1, \cdots, P_n$ are independent tail-area probabilities arising from n continuous distributions and $\lambda_1, \cdots, \lambda_n$ are unequal weights. The same distribution holds for linear combinations of independent chi-squareds, provided each has an even number of degrees of freedom. *S. W. Nash* (Vancouver, B.C.).

Romanovskiĭ, V. I. Quantile criteria for membership. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 3–11. (Russian)
The author proposes a test for the hypothesis that a sample $X_1, X_2, \cdots, X_n$ comes from a population with completely specified distribution function $F(x)$. The quantiles $q_1, \cdots, q_r$ of order r of $F(x)$ are then known and divide the range of $F(x)$ into r intervals $(-\infty, q_1)$, (q_1, q_2), $\cdots$, (q_{r-1}, ∞). If n observations are taken, they will be distributed over $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_s$ of these intervals in such a manner that α_j of the intervals contain n_j observations. In this way each sample gives a partition $(n_1^{\alpha_1} \cdots n_s^{\alpha_s})$ of the number n. The probability of obtaining a certain partition can be determined and is independent of $F(x)$; a short table of these probabilities, containing values for $r=4, 5, 10$ and $n=3, 4, 5$ is given. The hypothesis is rejected if the probability of the partition actually observed is too small. Alternative hypothesis and the power of the test are not discussed.
E. Lukacs (Washington, D.C.).

Romanovskiĭ, V. I. Quantile criteria for testing the hypothesis that a sample comes from a specified population. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 12–15. (Russian)
Let $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ be an ordered sample taken from a population with continuous distribution function. The author proposes the following test procedure for the hypothesis H_0 that the population distribution function is a completely specified distribution function $F(x)$. The observations are transformed according to $p_j = F(X_j)$ ($j=1, 2, \cdots, n$); these values can be considered as an ordered sample from a rectangular population over the interval (0, 1). The observed sample range is $r = p_n - p_1$. Let $P(R \leqq x)$ be the probability that the range R of a sample of size n taken from such a rectangular population should not exceed r; then $P(R \leqq x) = nx^{n-1} - (n-1)x^n$. The author rejects the hypothesis H_0 if $P(R \leqq r)$, where $r = p_n - p_1$ is too small. The power of the test is not discussed. *E. Lukacs* (Washington, D.C.).

Barton, D. E. A form of Neyman's ψ_K^2 test of goodness of fit applicable to grouped and discrete data. Skand. Aktuarietidskr. **38** (1955), 1–16.
The test of the title is based on any of many sets of orthogonal polynomials. In the case of equiprobable grouping the test reduces to that of Scott [Astrophys. J. **109** (1949), 194–207]. Such a grouping (stated to be asymptotically best) cannot be achieved in some discrete cases, to which the authors' test applies [see also David and Johnson, Biometrika **37** (1950), 42–49; MR **12**, 115]. Tables of moments and power are given. *J. Kiefer.*

van der Vaart, H. R. A closed expression for certain probabilities in Wilcoxon's two sample test. Experientia **12** (1956), 14–15.
Let U stand for the number of (x_i, y_j)-pairs, $i=1, \cdots, n_1, j=1, \cdots, n_2$ for which $x_i < y_j$. The paper gives formulas (without proof) for Pr $(U=u)$ and Pr $(U \leqq u)$ in the form of determinants, the probabilities being computed under the assumption that the x_i and y_j are random observations from the same (continuous) population. The formulas are

based on work by F. Brioschi [Ann. Sci. Mat. Fis. 7 (1856), 303–312] in partition theory. *G. E. Noether.*

Fraser, D. A. S.; and Guttman, Irwin. Tolerance regions. Ann. Math. Statist. **27** (1956), 162–179.

A necessary and sufficient condition for a tolerance region, i.e., a statistic whose values are sets in the space of a single observation, to be distribution-free is that the moments of its set characteristic function be independent of the underlying distribution. Using this it is shown that there are no non-trivial non-randomized symmetric distribution-free tolerance regions when the observations are real-valued, possibly with discrete distribution. Also for real-valued observations with absolutely continuous [with respect to Lebesgue measure] distributions, the non-randomized symmetric one-sided distribution-free tolerance regions have a simple characterization.

Using the set characteristic function representation of tolerance regions the problem of finding optimum distribution-free tolerance regions with specified expected coverage is formulated as a problem in testing of hypotheses. For several univariate and multivariate normal situations, optimum tolerance regions are found.

I. R. Savage (Stanford, Calif.).

Kemperman, J. H. B. Generalized tolerance limits. Ann. Math. Statist. **27** (1956), 180–186.

A method for constructing nonparametric tolerance limits due to Fraser [same Ann. **24** (1953), 44–55; MR **14**, 889] is generalized by allowing that each step of the construction may depend not only on the blocks previously formed but also on all the known boundary observations and, moreover, on certain sets of indices. The method of construction is presented clearly. Furthermore, Tukey's [ibid. **19** (1948), 30–39; MR **9**, 453] lexicographical ordering is replaced by a more general type of ordering.

I. R. Savage (Stanford, Calif.).

Ėidel'nant, M. I. Round-off errors. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 63–68. (Russian)

The author considers the effect on the mean and dispersion of a sample of grouping the sample data into class intervals of equal widths, and replacing each datum by the center of its class interval. Assuming a uniformly distributed population in each class interval, he calculates 99-per-cent probabilistic bounds for the errors of the sample mean and sample dispersion. The analytical tool is a refined form of Čebyšev's inequality, due to S. N. Bernštein [Theory of probability, Gos. Izdat., Moscow-Leningrad, 1927, p. 162]. There are some elementary applications. *G. E. Forsythe* (Los Angeles, Calif.).

Sarkadi, K. On Sheppard's correction concerning the mean deviation. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 183–189 (1955). (Hungarian. Russian and English summaries)

The author derives corrections, analogous to Sheppard's correction, for the mean deviation. He assumes that the grouping intervals (of length h) are located randomly and that the center of the interval containing the origin has a uniform distribution over $(-\frac{1}{2}h, \frac{1}{2}h)$. *E. Lukacs.*

Kamalov, M. K. Condition for independence of ratios of quadratic forms. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 88–91. (Russian)

Let $X_1, X_2, \cdots, X_n$ be n normally and independently distributed variables with zero means and common variance. Denote by $Q_1, Q_2, \cdots, Q_k$ k quadratic forms in the $X_1, \cdots, X_n$ and by $T_1=Q_1/Q_2$, $T_2=Q_3/Q_4 \cdots$ their ratios. The author shows that the random variables $T_1, \cdots, T_p$ are independently distributed provided that (i) $Q_1+Q_2+\cdots+Q_k=X_1^2+X_2^2+\cdots+X_n^2$ and (ii) $r_1+r_2+\cdots+r_k=n$, where r_j is the rank of Q_j. Several statistical applications of this result are also mentioned.

E. Lukacs (Washington, D.C.).

Laga [Laha], R. G. On stochastic independence of a homogeneous quadratic statistic and of the mean. Vestnik Leningrad. Univ. **11** (1956), no. 1, 25–32. (Russian)

The author investigates populations which have the following properties: (i) the population distribution function $F(x)$ has finite second moments; (ii) there exists a homogeneous quadratic statistic which is independently distributed of the sample mean. The first theorem of the paper is related to a characterization of the normal population and is a somewhat weakened version of an earlier result of the author [Biometrika **40** (1953), 228–229; MR **15**, 725]. The second theorem states that for the independence of $\sum_i a_{ii}x^2_i$ ($\sum_i a_{ii}\neq 0$) from the sample mean it is necessary and sufficient that $F(x)$ is a step function with two steps, located symmetrically about the origin. The discussion shows that, apart from the degenerate distribution, this step function and the normal distributions are the only population distribution functions admitting a homogeneous statistic which is independent of the sample mean. The author's work generalizes an earlier result of the reviewer concerning the independence of a homogeneous and symmetric quadratic statistic from the sample mean [Ann. Math. Statist. **23** (1952), 442–449; MR **14**, 297] by dropping the assumption of symmetry but maintaining the general course of the argument. *E. Lukacs* (Washington, D.C.).

Komarov, A. D. On a question of the application of mathematical statistics in working up the results of measurements of essentially positive quantities. Leningrad. Inžen.-Ėkonom. Inst. Trudy **1953**, no. 6, 218–225. (Russian)

Let x be normal with mean μ and standard deviation σ and set $w=|x|$. The paper indicates how μ and σ can be found in terms of $E(w)$ and $\sigma(w)$ and provides a table which facilitates the necessary computations. The problem of finding an upper limit for w which is not surpassed except with a given probability is also discussed.

G. E. Noether (Boston, Mass.).

Rosenblatt, Murray. On the estimation of regression coefficients of a vector-valued time series with a stationary residual. Ann. Math. Statist. **27** (1956), 99–121.

"Time series which are realizations of a vector-valued stochastic process of dimension two with a stationary disturbance are considered. Linear estimates of the regression coefficients of the time series are discussed, in particular the least-squares or classical estimate and the Markov estimate. The least-squares estimate is the estimate computed under the assumption that the components of the disturbance are orthogonal processes and orthogonal to each other. It is known that the Markov estimate is in general better than the least-squares estimate. The asymptotic behavior of the covariance matrices of the least-squares estimate and of the Markov

estimate is described. Conditions under which the least-squares estimate is as good asymptotically as the Markov estimate are obtained, that is, conditions under which the least-squares estimate is efficient asymptotically in the class of linear unbiased estimates. The analogues of the results described for vector-valued time series of dimension greater than two can be seen to hold." (Author's summary.) *J. Wolfowitz* (Ithaca, N.Y.).

Bates, Grace E. Joint distributions of time intervals for the occurrence of successive accidents in a generalized Pólya scheme. Ann. Math. Statist. **26** (1955), 705–720.

Let $P_{m,n}(T_1, T_2)$ be the conditional probability that an individual will incur n accidents during the time interval (T_1, T_2), given that he incurred m accidents during $(0, T_1)$, all accidents being non-fatal. The scheme considered has the following properties. (i) $P_{m,n}(T_1, T_2)$ does not depend on when exactly the last-mentioned m accidents occurred, (ii) $P_{m,n}(T_1, T_1)=0$ for every m and for $n\geqq 1$, and

$$\text{(iii)} \qquad \frac{\partial}{\partial T_2} P_{m,n}(T_1, T_2)|_{T_2=T_1} = \frac{\gamma\lambda_m}{1+\nu T_1},$$

where $\nu\geqq 0$, the λ's are arbitrary positive numbers and $\gamma=-1$, 1 and 0 when $n=0$, 1 and >1, respectively. Given m and n the author derives the joint distribution of $\tau_1, \cdots, \tau_n$, where τ_i is the time from T_1 to the occurrence of the ith accident in (T_1, T_2). For the particular case when $\lambda_{m+k+1}-\lambda_{m+k}=\psi$ $(k=0, 1, \cdots, n-1)$, and $\nu=0$ the author constructs a uniformly most powerful unbiased test of the hypothesis $\psi=0$ against alternatives $\psi\neq 0$. The data used in the test are τ's collected during (T_1, T_2) on several groups of individuals, all with the same ψ and the rth group consisting of all individuals with $n=r$. The test statistic is the grand mean of all τ's observed. The power function of the test is determined approximately.

D. M. Sandelius (Göteborg).

Parzen, Emanuel. On consistent estimates of the spectral density of a stationary time series. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 154–157.

The estimates of the title are obtained for processes which are stationary of order 4 and whose covariance functions are in $L_1 \cap L_2$. The estimates are obtained by using averages of the periodogram with respect to various averaging kernels, generalizing the method used by Bartlett [Biometrika **37** (1950), 1–16; MR **12**, 35] and others. The order of convergence is discussed. No proofs.

J. Kiefer (Ithaca, N.Y.).

Patnaik, P. B. Hypotheses concerning the means of observations in normal samples. Sankyā **15** (1955), 343–372.

Let $X_1, \cdots, X_n$ be independent normal random variables with common unknown variance σ^2. Let $\mu_i=EX_i$. Among others, the author considers the following problems in testing hypotheses and in each case finds a class of similar tests and the best (or best unbiased) member of the class: (1) H_0: $\mu_i/\sigma=\varrho_i$ (specified), H_1: $\mu_i/\sigma=c\varrho_i$, where c may be specified to be >1, <1, or $\neq 1$; in the case where all ϱ_i are equal the test reduces to the non-central t-test considered by Johnson and Welch [Biometrika **31** (1940), 362–389; MR **1**, 346] and others, while in the general case both numerator and denominator of the test criterion are non-central; (2) assuming $\mu_i/\sigma=\varrho$ for $n-m$ of the X_i (it is not known for which) and $=\varrho+\gamma$ for the other m (where γ is known), H_0: $\varrho=\varrho_0$ (specified), H_1: $\varrho>\varrho_0$ or $\varrho<\varrho_0$ or $\varrho\neq\varrho_0$. *J. Kiefer* (Ithaca, N.Y.).

Whittle, P. On principal components and least square methods of factor analysis. Skand. Aktuarietidskr. **35** (1952), 223–239 (1953).

Suppose we have p measurements on each of q persons yielding the matrix of scores $X=(x_{ij})$, $i=1, 2, \cdots, p$, $j=1, 2, \cdots, q$. Factor analysis seeks to explain this complex of measurements by r factors say, so that

$$x_{ij}=\alpha_{i1}\beta_{j1}+\alpha_{i2}\beta_{j2}+\cdots+\alpha_{ir}\beta_{jr}+e_{ij},$$

where β_{jk} is the value of the kth factor for the jth person, α_{ik} the factor loading in the ith measurement and e_{ij} is the residual error. The author considers the estimation of the factor loadings and factor values by minimizing the residual sum of squares

$$\sum_i \sum_j (x_{ij}-\sum_{k=1}^{r} \alpha_{ik}\beta_{jk})^2,$$

assuming that the variates e_{ij} are independent, with zero means and equal variances. Let a_k and b_k denote respectively the least-squares estimates of $(\alpha_{1k}, \alpha_{2k}, \cdots, \alpha_{pk})$ and $(\beta_{1k}, \beta_{2k}, \cdots, \beta_{qk})$; then (1) a_k is that eigenvector of XX' corresponding to the kth largest eigenvalue of the matrix, l_k, so normalised that $a_k'a_k=l_k/q$; (2) b_k is the corresponding eigenvector of $X'X$ so normalised that $b'_kb_k=q$; (3) the different a_k's are mutually orthogonal, as are the different b_k's; (4) the reduction in residual sum of squares due to the introduction of r common factors is $\sum_{k=1}^{r} l_k$. The author also discusses the associated testing problems and evaluates the asymptotic variances and covariances of the loading estimates. *S. Kullback.*

Bainbridge, J. R.; Grant, Alison M.; and Radok, U. Tabular analysis of factorial experiments and the use of punch cards. J. Amer. Statist. Assoc. **51** (1956), 149–158.

Nash, Stanley W. Contribution to the theory of experiments with many treatments. Univ. California Publ. Statist. **2** (1956), 167–183.

In successive experiments involving many treatments the following situation may arise. A first experiment, testing m treatments, will have probability near one of showing a significant treatment effect, but a second experiment, testing the r treatments appearing highest and the r appearing lowest in the first trial, will have probability only slightly above the significance level of showing a significant treatment effect. It is shown that this may happen even when the treatment effects are drawn at random from a normally distributed population of effects, and measurement errors are distributed normally and independently. Detailed calculations are given for balanced designs. *P. Meier* (Baltimore, Md.).

Tocher, K. D. The design and analysis of block experiments. J. Roy. Statist. Soc. Ser. B. **14** (1952), 45–91; discussion 91–100.

A systematic account of the analysis and design of block experiments, based on the Gauss-Markoff theorem, is developed in matrix notation. Many well known results are presented in concise form. Typical of the new results are the following: Let there be t treatments arranged in b blocks such that each block contains k experimental units and the ith treatment is replicated r_i times. The mean response of the ith treatment in the jth block is t_i+b_j. If the experiment requires that the variance of the difference between the ith and i'th treatment estimates be

$2\theta_{ii'}$ ($\theta_{ii}=0$), then a necessary condition on the $t\times b$ incidence matrix $\mathbf{n}$ is

$$\mathbf{nn}'=k(\mathbf{r}^{\delta}-\boldsymbol{\theta}^{-1})+\frac{k}{\mathbf{1}'\boldsymbol{\theta}^{-1}\mathbf{1}}\boldsymbol{\theta}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\theta}^{-1},$$

where $\mathbf{r}^{\delta}$ denotes the diagonal matrix with elements $(r_1, r_2, \cdots, r_t)$ on the main diagonal, $\mathbf{1}=t\times 1$ column vector whose elements are all unity, and $\boldsymbol{\theta}=\|-\theta_{ii'}\|$, $(i, i'=1, 2, \cdots, t)$.

The author introduces balanced ternary designs, equigroup balanced designs, and mixed group balanced designs. The ternary designs allow a treatment to occur either 0, 1, or 2 times in any block and have the property that $\theta_{ii'}=\theta$ for all $i\neq i'$. The equi-group designs are such that the treatments can be divided into m groups of t treatments each, so that comparisons within the same group have a smaller variance than comparisons between different groups. These are special cases of group divisible designs [Bose and Connor, Ann. Math. Statist. **23** (1952), 367–383; MR **14**, 124; Bose and Shimamoto, J. Amer. Statist. Assoc. **47** (1952), 151–184; MR **14**, 67]. The mixed group balanced designs are characterized by having unequal groups of treatments and have the same property with respect to the variance between two treatment estimates as the equi-group designs. Examples of each of these three classes of designs are also given. *M. Zelen.*

Gervaise, Anne-Marie. Sur l'interprétation des théorèmes de Mood. C. R. Acad. Sci. Paris **242** (1956), 984–986.

In order that, for all probability measures P on a space D for which $0<P(E)<1$ (where $E\subset D$), $\int_E f dP>P(E)\int_D f dP$ where f is a bounded positive function, it is clearly necessary and sufficient that $\inf_E f>\sup_{D-E} f$. This fact is used to explain a result of Mood on sampling inspection [Ann. Math. Statist. **14** (1943), 415–425; MR **5**, 210]. *J. Kiefer* (Ithaca, N.Y.).

Békéssy, A. Über der Wahrscheinlichkeitsverteilung der Impulsanzahl bei fehlerhaft arbeitenden Untersetzern. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 171–181 (1955). (Hungarian. Russian and German summaries)

A scaler is an instrument connected to a counter which permits the counter to register only every kth signal where k is a power of 2. It consists of a number of stages each of which transmits only every other signal to the next stage. In the paper it is assumed that each stage transmits an incoming signal only with a certain probability so that the counter may lose a number of counts due to malfunctioning of the scaler. Several reasonable assumptions are made concerning the incoming signal and the operation of the counter and the probability that the counter is registering n counts in time t is determined. The mathematical tools are Laplace transforms and generating functions. Moments of the distribution of counts are determined and are used to devise a check for the reliability of the instrument. *E. Lukacs.*

Kutaĭ, A. K. Study of the precision of production with an application of the distribution laws of essentially positive quantities. Vestnik Leningrad. Univ. **11** (1956), no. 1, 101–103. (Russian)

Kamalov, M. K. Statistical control of production processes taking account of time. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 74–87. (Russian)

The author studies a production process and is primarily interested in the non-random factors affecting the quality of the product. *E. Lukacs* (Washington, D.C.).

Kordonskiĭ, H. B. Application of the theory of Markov chains to control of lots. Vestnik Leningrad. Univ. **10** (1955), no. 11, 75–78. (Russian)

A k-phase acceptance sampling procedure is considered, the rules of transition from one phase to another phase being those of a Markov-chain process. Formulae for the average outgoing quality and amount of inspection are given. Two examples are considered. *G. E. Noether.*

Medgyessy, P.; Rényi, A.; Tettamanti, K.; and Vincze, I. Mathematical investigation of chemical countercurrent distribution, in case of non-complete diffusion. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 81–97 (1955). (Hungarian. Russian and English summaries)

The chemical process mentioned in the title serves to determine whether a certain substance is pure or a mixture. It is carried out by letting the substance diffuse from one solvent into another which does not mix with the first solvent. The process is carried consecutively through a number of cells. If the substance is pure and if equilibrium is reached in each stage then the resulting distribution of the solute in the cells is binomial. The authors consider the case where each stage of the process is interrupted before the equilibrium state is reached and determines the distribution of the solute by means of generating functions. A probabilistic model which considers the random walk performed by the molecules of the substance is also discussed. *E. Lukacs* (Washington, D.C.).

Peretti, Jean. Définition et méthode de calcul de la fonction de répartition statistique attachée à une grandeur physique. C. R. Acad. Sci. Paris **242** (1956), 1416–1417.

See also: Medgyessy, p. 862; Takács, p. 866; Archbold, p. 886; Ermilov, p. 902.

Theory of Games, Mathematical Economics

★ **Heller, I. On the travelling salesman's problem.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 643–665. National Bureau of Standards, Washington, D.C., 1955.

Section 1 of the paper introduces notation and terminology and summarizes some of the basic known results. Section 2 deals with the special case where the matrix of distances is symmetric, and some partial results are obtained for this case. Section 3 deals with neighbor relations on the convex polyhedron P spanned by the $n\times n$ permutation matrices. Two vertices p_1 and p_2 of P are called neighbors of order k if they have a face of dimension k but no face of lower dimension in common. The principal result states that p_1 and p_2 are neighbors of order k if and only if $p_1^{-1}p_2$ is a product of k disjoint cycles (not counting cycles of length 1). Proofs are given in the final section. *D. Gale* (Providence, R.I.).

★ **Gass, Saul I. A first feasible solution to the linear programming problem.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 495–508. National Bureau of Standards, Washington, D.C., 1955.

The author proposes a method for obtaining a first

feasible solution to the linear-programming problem. The purpose of the method is to obtain a first feasible solution which is closer to the minimizing solution than that obtained by the usual procedure. This should then have the effect of reducing the number of iterations necessary to arrive at a minimizing solution. *D. Gale.*

Antosiewicz, H. A. Analytic study of war games. Naval Res. Logist. Quart. 2 (1955), 181–208.

Solution of a war-game model, attributed to an Army officer, consisting of a set of six non-linear ordinary differential equations of first order, which express attrition rates according to Lanchester-type laws, the variables representing the strengths of the offensive and defensive forces, and the rates of production, of each of the two opponents. *L. Gillman* (Lafayette, Ind.).

Gallego Díaz, José. A new metric as an attempt to axiomatize economics. Theoria 2 (1954), no. 7–8, 65–70. (Spanish)

TOPOLOGICAL ALGEBRAIC STRUCTURES

Kapuano, Isaac. Sur les ensembles boréliens représentant des corps de nombres complexes. C. R. Acad. Sci. Paris 240 (1955), 2193–2196.

Using hereditarily indecomposable continua, the author gives examples of fields of complex numbers, representing Borel sets of classes 2, 3, 4 and 5. *E. E. Moise.*

Topological Groups

Takahashi, Reiji. Un théorème de commutation. C. R. Acad. Sci. Paris 242 (1956), 1103–1106.

Let K_1 and K_2 be compact subgroups of the locally compact group G such that $K_1 \subseteq K_2$. Let α be a continuous homomorphism of K_2 into the complex numbers of modulus one and let H^α be the subspace of $L^2(G)$ consisting of all functions f in $L^2(G)$ such that $f(k_1 g k_2) = \alpha(k_1)\alpha(k_2)f(g)$ for all k_1 in K_1 and all k_2 in K_2. For $i=1, 2$ let A_i^α be the self-adjoint subalgebra of $L^1(G)$ consisting of all summable functions ϕ such that $\phi(k_i g k_i') = \alpha(k_i)\alpha(k_i')\phi(g)$ for all k_i and k_i' in K_i. Then H^α is invariant under convolution from the left with members of A_1^α and under convolution from the right with members of A_2^α. Let $\mathcal{L}^\alpha$ and $\mathcal{R}^\alpha$ denote the rings of operators (in the sense of Murray and von Neumann) generated by these two families of convolution operators. This note is devoted to proving that $\mathcal{L}^\alpha$ and $\mathcal{R}^\alpha$ are the full commutators of one another. This theorem is well known when $K_1 = K_2 = (e)$ and the author points out that this classical case of the theorem readily implies the more general case in which K_1 is a normal subgroup of G. *G. W. Mackey.*

★ **Singer, I. M. Report on group representations.** Report of an international conference on operator theory and group representations, Arden House, Harriman, N. Y., 1955, pp. 11–26. Publ. 387, National Academy of Sciences—National Research Council, Washington, D. C., 1955.

Expository paper.

Lie Groups, Lie Algebras

Tits, J. Sur certaines classes d'espaces homogènes de groupes de Lie. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° 29 (1955), no. 3, 268 pp.

The main purpose of this paper is to study geometric properties of semi-simple Lie groups. The paper contains such a wealth of detailed information concerning concrete examples of Lie groups and Lie algebras that it would be impossible to reproduce these results in this review. We will therefore limit ourselves to summarizing the general theorems and will try to indicate whenever possible what type of specific information is contained in this paper.

Part one is devoted to a resumé of the theory of semi-simple algebras over the complex and real fields. The concept of regular subalgebra introduced here is fundamental for the rest of the paper. A subalgebra H of a Lie algebra G is called regular if there exists a Cartan subalgebra C such that one can find a base for H, as a vector space, consisting of elements of C and proper vectors of G relative to C. Certain particularly important examples of regular subalgebras may be obtained as follows: Let Σ be a system of simple roots of G and Σ' any sub-system (a slightly more general case is considered in the paper). Let $G^{ss}(\Sigma')$ be the subalgebra of G generated by the roots belonging to Σ' and the integer linear combination of the proper vector belonging to the roots of Σ'. Let Σ'^+ be the system of all positive roots of G whose coordinates relative to Σ' are not zero. Let $\langle\Sigma'\rangle^*$ be the linear space of C orthogonal, by means of the Killing form, to Σ'. Let $G^{nilp}(\Sigma')$ be the nilpotent subalgebra of G generated by the proper vectors corresponding to the roots of Σ'^+. Define

$$G^{reg}(\Sigma') = G^{nilp}(\Sigma') \cdot \langle\Sigma'\rangle^*,$$

where the dot denotes the subalgebra of G generated by products of pairs of elements, one from G^{nilp} and one from $\langle\Sigma'\rangle^*$. The author defines

$$G(\Sigma') = G^{ss}(\Sigma/\Sigma') \cdot G^{reg}(\Sigma'),$$

where Σ/Σ' denotes the simple roots of G which do not belong to Σ'. In this section is also an outline of representation theory in terms of projective groups of motions.

Part II of the paper is devoted to an exposition of the following topics: 1. Homogeneous spaces; 2. certain facts concerning projective spaces over the reals, complex numbers, and quaternions; 3. special classes of spaces.

In his treatment of homogeneous spaces the author introduces the following two concepts. 1. Locally homogeneous spaces. Let G and G' be a Lie group and H and H' be subgroups of G and G' respectively. Then the author says that the pair G, H is locally isomorphic to the pair G', H' if there exists a neighborhood of the identity U and U' in G and G', respectively, and a local isomorphism φ of U onto U' such that $\varphi(U \cap H)$ is mapped into $U' \cap H'$. Then the homogeneous space G/H is called locally isomorphic to the homogeneous space G'/H'. 2. Real form of a homogeneous space. Let $\tilde{G}$ be a complex analytic group (connected Lie group), $\tilde{H}$ an analytic subgroup and σ an involutive anti-automorphism of $\tilde{G}$ leaving $\tilde{H}$ invariant. Let G and H be the subgroups of $\tilde{G}$ and $\tilde{H}$, respectively, which are left point-wise fixed by σ. Then G and H are called the real forms of $\tilde{G}$ and $\tilde{H}$, respectively, and the space $E = G/H$ is called the real form of $\tilde{E} = \tilde{G}/\tilde{H}$; $\tilde{E}$ is called the complexification of E. It is possible for a real homogeneous space to have no complexifications or to

have many. Complexification is always defined and unique for the equivalence classes of locally homogeneous spaces.

In his study of projective spaces the author discusses involutions (biaxial and non-biaxial), reciprocities and polarities over the reals, complex numbers and quaternions and gives the determination of all of the above up to projective equivalence. He further discusses commuting polarities and involutions and classifies such couples. The study of special spaces includes a discussion of Grassmann spaces, mixed Grassmann spaces, Cayley spaces, quadric spaces and the octanes of Cayley. He shows in particular that all projective spaces, quadric spaces or Grassmannians of projective spaces or quadric spaces are spaces of the form $G/G(\Sigma')$ or a real form of such a space, where G is a classical group.

The purpose of Part III is to study the homogeneous spaces G/H, where G is a complex semi-simple Lie group and H is a subgroup with Lie algebra of the form $G(\Sigma')$, where Σ' is a system of simple roots of G. $G/G(\Sigma')$ will be denoted by $G[\Sigma']$. Some of the more general results listed or proved by the author are the following. 1. Every space G'/H' which is locally isomorphic to $G/G(\Sigma')$ is isomorphic to $G/G(\Sigma')$. 2. The only subgroup of G having $G(\Sigma')$ as Lie algebra is the connected subgroup with this property. 3. $G[\Sigma']$ is compact.

Given two systems of roots Σ' and Σ'', we may consider them as vertices of the Schläfli diagram. The author says that Σ'' is reductive with respect to Σ' (or mod Σ') if every vertex α of Σ'' can be connected to a vertex of Σ' by a chain none of whose elements except α belong to Σ''. The author says that Σ''' is intermediate between Σ' and Σ'', $\Sigma'<\Sigma'''<\Sigma''$, if every chain connecting a vertex of Σ'' to a vertex of Σ' contains a vertex of Σ'''. Now, given two systems Σ'' and Σ', we can decompose Σ'' in one and only one way into the union, $\Sigma''=\Sigma_1''\cup\Sigma_2''\cup\cdots$ of disjoint systems, each of which is reductive mod Σ' and such that $\Sigma'<\Sigma_i''<\Sigma_{i+1}''$. Then Σ_1'' is called the reduction of Σ'' with respect to Σ'.

In the space $G[\Sigma']$ consider the point $\mathfrak{p}$ which is the image of $G(\Sigma')$. The set of transforms E of the point $\mathfrak{p}$ by a subgroup of the form $G(\Sigma'')$ depends only on the reduction in Σ_1'' of Σ'' with respect to Σ'. Here E and its transforms by the elements of G will be called the Σ_1'' planes of $G[\Sigma']$. The space of Σ_1'' planes of $G[\Sigma']$ is (canonically) isomorphic to the space $G[\Sigma_1'']$.

The author then discusses the concept of incidence. Let Σ' be a system of simple roots of the group and let Σ'' and Σ''' be two other systems reductive mod Σ' and satisfying the relation $\Sigma'<\Sigma'''<\Sigma''$. The points of the space $G[\Sigma'']$ and $G[\Sigma''']$ represent respectively the Σ'' planes and the Σ''' planes of $G[\Sigma']$. Let $\mathfrak{p}\in G[\Sigma'']$ and $\mathfrak{q}\in G[\Sigma''']$. Let $\mathfrak{p}''$ and $\mathfrak{q}'''$ denote the image of $G(\Sigma'')$ and $G(\Sigma''')$ under the induced mapping. Then a necessary and sufficient condition for the plane representing $\mathfrak{p}$ to contain the plane representing $\mathfrak{q}$ is that there exist an element of G which simultaneously maps $\mathfrak{p}$ to $\mathfrak{p}''$ and $\mathfrak{q}$ to $\mathfrak{q}'''$. Under the above conditions we will say that $\mathfrak{p}$ and $\mathfrak{q}$ are incident. Then, given any two systems of simple roots Σ' and Σ'', the points of $G[\Sigma']$ which are incident to a given point of the space $G[\Sigma'']$ is a Σ_1'' plane, where Σ_1'' is the reduction of Σ'' mod Σ' and conversely. These notions are then generalized to include arbitrary $G[\Sigma'']$ and $G[\Sigma''']$.

The author also poses the following problem: Assume we are given two semi-simple groups G and H, a system Σ' of simple roots of G and a system Π' of simple roots of H; determine all the representations of the space $H[\Pi']$ into the space $G[\Sigma']$. The author solves this question for the cases where $G[\Sigma']$ is a projective space and where $G[\Sigma']$ is the Grassmannian of lines in a projective space. The author then applies these general result to study the exceptional Lie groups.

Due to the detailed nature of the results of Part IV and the fact that many of the results have been announced earlier, we will limit ourselves to giving the author's introduction to this part.

"The purpose of this fourth part is essentially the solution of the following problems: The determination of all the transitive groups of automorphisms of the real, complex, quaternion projective spaces. The determination of the homogeneous spaces G/H of Lie groups which are isotropic, that is to say, those for which G operates transitively on the contact elements of G/H. The demonstration of previously announced results [Bull. Soc. Math. Belg. **1952**, 44–52; MR **15**, 334] and in particular, the determination of all locally compact metric spaces which are two-point homogeneous. The determination of all the homogeneous spaces G/H of Lie groups such that G operates transitively on the couples, the triples, $\cdots$, the n-tuples of points of G/H. More generally, to determine all the locally compact transformation groups of a not totally disconnected space, which are doubly, triply, $\cdots$, n-tuply transitive. For $n=2$ the last problem is only solved for the dimension of the space greater than 124."

L. Auslander (Princeton, N.J.).

Dobrescu, Andrei. Relations entre les tenseurs associés à un groupe G_r. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 39–42. (Romanian. Russian and French summaries)

The constants of structure C_{kl}^{h} of a continuous group G_r can be considered as the components of a constant tensor in the parameter space with respect to linear and homogeneous transformations of the parameters. Following Vranceanu who has introduced $C_{h}=C_{hz}^{z}$ and $C_{hk}=C_{h\beta}^{\alpha}C_{k\alpha}^{\beta}$, the author considers the tensor $C_{hkl}=C_{h\beta}^{\alpha}C_{k\gamma}^{\beta}C_{l\alpha}^{\gamma}$ and proves a number of theorems connecting the properties of these tensors with the properties of G_r. *R. Blum.*

Naĭmark, M. A. Continuous analogue of Schur's lemma and its application to Plancherel's formula for the complex classical groups. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 3–16. (Russian)

This paper gives a detailed account of results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 185–188; MR **16**, 597]. However the quotation of results in the indicated review needs to be corrected as follows: In the definition of the ring R the set of all bounded linear operators on H should be replaced by the set of all completely continuous operators on H. *G. W. Mackey.*

Naĭmark, M. A. Description of all irreducible unitary representations of the classical groups. Amer. Math. Soc. Transl. (2) **2** (1956), 141–145.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 883–886; MR **14**, 16.

Gel'fand, I. M.; and Šapiro, Z. Ya. Representations of the group of rotations of 3-dimensional space and their applications. Amer. Math. Soc. Transl. (2) **2** (1956), 207–316.

Translated from Uspehi Mat. Nauk (N.S.) **7** (1952), no. 1(47), 3–117; MR **13**, 911.

Gel'fand, I. M.; and Graev, M. I. Unitary representations of the real unimodular group (principal nondegenerate series). Amer. Math. Soc. Transl. (2) **2** (1956), 147–205.

Translated from Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 189–248; MR **15,** 199.

Gluškov, V. M. Locally nilpotent groups without torsion, complete over simple topological fields. Mat. Sb. N.S. **37(79)** (1955), 477–506. (Russian)

The author proposes the following object: A topological group G which is simultaneously a topological Lie algebra (possibly infinite-dimensional) over a topological field F of characteristic 0 in such a way that the Campbell-Hausdorff series converges to the (group) product. One example is provided by taking G to be a free Lie algebra over F with the obvious topology, defining the group operation via Campbell-Hausdorff. For a second example, take G to be a locally nilpotent Lie algebra; then the Campbell-Hausdorff series always expires in a finite number of terms and there are no convergence problems. In the first three sections these examples are studied and generalized, uniqueness theorems are proved, etc. In § 4 the author starts anew with a topological group G where it is assumed that there are sufficiently many continuous homomorphisms of the additive group of F so as to cover G (this is the completeness over F referred to in the title); it is further assumed that F is simple (has no proper closed subfields). Note that G, as a consequence, is divisible. If one further assumes that G is locally nilpotent and torsion-free, it is shown that G is attached to a Lie algebra in the fashion described above. If F is the field of p-adic numbers, G is abelian and satisfies the second axiom of countability, then G is isomorphic to the direct sum of a finite number of copies of F; in the proof the author makes use of the structure theorem of Mackey [Bull. Amer. Math. Soc. **52** (1946), 940–944; MR **8,** 311]. It is noted that one cannot drop the second axiom of countability. *I. Kaplansky* (Chicago, Ill.).

Dixmier, J. Certaines factorisations canoniques dans l'homologie et la cohomologie des algèbres de Lie. J. Math. Pures Appl. (9) **35** (1956), 77–86.

Let L be a finite-dimensional Lie algebra over a field K of characteristic 0, let R denote the radical of L, and let M be a finite-dimensional L-module. It is known that the cohomology space $H^*(L, M)$ admits of a factorization: $H^*(L, M) \approx H^*(L/R, K) \otimes H^*(R, M)^L$, where the second factor is the subspace of $H^*(R, M)$ consisting of the cohomology classes annihilated by L. This factorization was derived from a more general theorem involving the relative cohomology space with respect to a reductive subalgebra [Hochschild and Serre, Ann. of Math. (2) **57** (1953), 591–603; MR **14,** 943] by using a Levi decomposition, or, equivalently, a maximal semisimple subalgebra of L. Here, the author uses the Malcev uniqueness theorem concerning the Levi decompositions in order to show that the above isomorphism is actually canonical, i.e., does not depend on the particular choice of the Levi decomposition. The main point is that there actually exists a canonical monomorphism $H^*(R, M)^L \to H^*(L, M)$ inverse to the restriction map.

Specializing to the case $M=K$, and dualizing, the author also obtains a canonical isomorphism of homology spaces: $H(L) \approx H(L/R) \otimes H(R)_L$. Since $H(L/R)$ has also the structure of an algebra (with identity) over K, this yields the structure of a unitary $H(L/R)$-module on $H(L)$. On the other hand, extending results of J. L. Koszul, the author shows that if S is any reductive subalgebra of L that is not homologous to 0 then the natural map $H(S) \to H(L)$, when effected by means of suitably chosen cycles, leads to the structure of an $H(S)$-module on $H(L)$ (derived from the exterior multiplication in the chain group). If S is taken to be a Levi component of L, this module structure of $H(L)$ induces the above canonical $H(L/R)$-module structure, via the natural isomorphism of S onto L/R.

G. P. Hochschild (Berkeley, Calif.).

Patterson, E. M. Bounds for the indices of nilpotent and solvable Lie algebras. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954–55), 200–208 (1956).

The results of a recent paper by the author [Proc. Cambridge Philos. Soc. **51** (1955) 37–40; MR **16,** 562] are extended and improved. We use the terminology of the review of the earlier paper, and state several typical theorems concerning any nilpotent Lie algebra A of dimension m, class s, and index r: (1) $r \leq [\log_2(s-1)]+1$; (2) $A^{(2)} \subseteq A^5$ is implied by any one of (a) $\dim A^2 = m-2$, (b) $s=m$, (c) $s=m-1$, or (d) characteristic $F \neq 2$ and $s=m-2$; (3) $r \leq 2$ if $m=3, 4, 5$, $r \leq 3$ if $6 \leq m \leq 10$, $r \leq 4$ if $11 \leq m \leq 20$, and $r \leq k+1$ if $2^k+k+1 \leq m \leq 2^{k+1}+k+1$ ($k \geq 4$). Examples are given to show that some of the results obtained are the best possible.

R. D. Schafer (Storrs, Conn.).

See also: Singer, p. 874.

Topological Vector Spaces

Warner, Seth. Weak locally multiplicatively-convex algebras. Pacific J. Math. **5** (1955), 1025–1032.

Let E be a real or complex linear algebra, E' a total subspace of the algebraic dual of E (as a linear space). The author proves that the weak topology $\sigma(E, E')$ on E determined by E' is locally m-convex [E. A. Michael, Mem. Amer. Math. Soc. no. **11** (1952); MR **14,** 482] if and only if, for all $g \in E'$, the kernel of g contains a weakly closed ideal of E, of finite co-dimension. A similar condition, necessary and sufficient for continuity of multiplication under $\sigma(E, E')$, is also obtained.

Next, let E be the algebra of polynomials, without constant term, in an indeterminate x; for scalar λ let f_λ be the multiplicative linear functional: $f_\lambda(x^r) = \lambda^r$. If E has a locally m-convex topology, let $\mathfrak{M}(E)$ be the set of all continuous multiplicative linear functionals (all of which must be of the form f_λ). Then the author shows that if L is any subset of the scalar field, containing zero, there is a weak locally m-convex topology on E such that $\mathfrak{M}(E)$ and L are homeomorphic under the mapping $f_\lambda \to \lambda$. Further results are obtained in case L is infinite.

J. H. Williamson (Princeton, N.J.).

Del Pasqua, Dario. Iperanaliticità dei funzionali analitici. Rend. Mat. e Appl. (5) **14** (1955), 594–601.

Il s'agit d'un complément à un travail précédent [mêmes Rend. (5) **12** (1953), 188–228; MR **15,** 966]. L'auteur démontre l'identité entre les fonctionnelles (resp. opérateurs) analytiques et les fonctionnelles (resp. opérateurs) hyperanalytiques. *J. Sebastião e Silva.*

Ehrenpreis, Leon. Analytic functions and the Fourier transform of distributions. I. Ann. of Math. (2) **63** (1956), 129–159.

L'auteur développe ici sa théorie de la transformation

de Fourier sur l'espace $\mathcal{D}'$ de toutes les distributions dans R^n [cf. Amer. J. Math. **76** (1954), 883–903; MR **16**, 834]. On considère d'abord la transformation de Fourier

$$\mathcal{F}[f(x)]=(2\pi)^{-n/2}\int e^{-ixz}f(x)dx \quad (x \in R^n,\ z \in C^n)$$

pour $f \in \mathcal{D}$. L'espace $\mathcal{F}(\mathcal{D})$, noté **D**, est constitué par les fonctions entières de type exponentiel sur C^n, à décroissance rapide sur R^n; sa topologie naturelle (image de celle de $\mathcal{D}$ par $\mathcal{F}$) est explicitée par l'auteur. Les éléments de **D'** ne sont plus, en général, des distributions. La transformation de Fourier $\mathbf{F}:\mathcal{D}'\to\mathbf{D}'$ est, par définition, la transposée de $\mathcal{F}^{-1}$, c.à.d. $\mathbf{F}(S)\cdot\varphi=S\cdot\mathcal{F}^{-1}(\varphi)$, pour $S \in \mathcal{D}'$, $\varphi \in \mathbf{D}$. Cela étant, il s'agit de généraliser à **F** la théorie de la transformation de Fourier, telle que Schwartz l'a développée pour le cas des distributions tempérées. La convolution est définie dans **D** par $(F*G)(z)=\int F(x)G(z-x)dx$; pour $F \in \mathbf{D}$, $S \in \mathbf{D}'$, on pose $(F*S)\cdot G=S\cdot(F*G)$, $FS\cdot G=S\cdot{}_{\oplus}FG$, où ${}_{\oplus}F(z)=\overline{F(-\bar{z})}$ $(G \in \mathbf{D})$, et on généralise les propriétés usuelles concernant ces deux notions de produit.

La dernière partie de ce travail est consacrée à une étude de la transformation de Fourier sur l'espace $\mathcal{E}$ de Schwartz et sur $\mathcal{E}'$. A cet effet, les éléments de $\mathcal{E}$ sont interprétés comme les distributions dont le produit multiplicatif par tout élément de $\mathcal{D}$ est encore un élément de $\mathcal{D}$ et, l'espace $\mathcal{E}$ étant considéré comme ensemble d'applications linéaires continues de $\mathcal{D}$ dans $\mathcal{D}$, on voit que sa topologie coïncide avec celle de la convergence compacte. De même, $\mathcal{E}'$ est identifié à l'espace des applications linéaires continues de $\mathcal{D}'$ dans $\mathcal{D}'$, permutant avec les dérivations. On adopte des critères analogues pour définir les topologies des espaces **E** et **E'**, images de Fourier resp. de $\mathcal{E}$ et $\mathcal{E}'$, de façon que **F** définisse des anti-isomorphismes topologiques $\mathcal{E}\to\mathbf{E}$, resp. $\mathcal{E}'\to\mathbf{E}'$. L'espace **E'** est formé par les éléments de **D'** qui se représentent par fonctions entières de type exponential à croissance lente sur R^n.

J. Sebastião e Silva (Lisbonne).

Ehrenpreis, Leon. The division problem for distributions. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 756–758.

Ehrenpreis, Leon. Completely inversible operators. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 945–946.

Dans un travail précédent [Amer. J. Math. **76** (1954), 883–903; MR **16**, 834] l'auteur avait démontré que toute équation aux dérivées partielles $DS=T$ admet une solution S, pour toute distribution T d'ordre fini. Maintenant, dans le premier de ces articles, il généralise ce résultat au cas d'une distribution T quelconque et au cas où D est un polynôme mixte d'opérateurs de dérivation et d'opérateurs de différences finies.

Dans le deuxième article, l'auteur considère, plus généralement, les équations de convolution $W*S=T$, où W est une distribution à support compact; W est dite complètement inversible, si cette équation a une solution S pour toute distribution $T \in \mathcal{D}'$. On a le résultat suivant: pour que W soit complètement inversible, il faut et il suffit que l'image de Fourier, $w=F(W)$, de W, soit à décroissance lente, c.à.d. qu'il existe trois nombres positifs a, b, c, tels que, pour tout $x \in R^n$, on puisse trouver $y \in R^n$, de façon que $|x-y|\leq a\log(1+|x|)$, $w(y)\geq b/(1+|y|^c)$. On établit, d'ailleurs: pour que W soit complètement inversible, il suffit que l'équation $W*S=\delta$ ait une solution S (solution élémentaire); cela répond à une question posée, et partiellement résolue, par L. Schwartz. L'auteur indique d'autres conditions nécessaires et suffisantes pour que W soit complètement inversible: a) $W*\mathcal{E}=\mathcal{E}$; b) $W*f\to f$ est une application continue de $W*\mathcal{D}$ dans $\mathcal{D}$, etc.

J. Sebastião e Silva (Lisbonne).

Sebastião e Silva, J. Sur une construction axiomatique de la théorie des distributions. Gaz. Mat., Lisboa **15** (1954), no. 59, 6–10.

Summary of results in Univ. Lisboa. Rev. Fac. Ci. A. **4** (1955), 79–186; **5** (1956), 169–170 [MR **17**, 766].

Banach Spaces, Banach Algebras

Grothendieck, A. Une caractérisation vectorielle-métrique des espaces L^1. Canad. J. Math. **7** (1955), 552–561.

Un espace de Banach C possède la propriété de Nachbin ou est un espace (N), si toute application linéaire continue u, dans C, d'un sous-espace vectoriel F d'un espace de Banach E, est prolongeable en une application linéaire de même norme de E dans C. Nachbin [Trans. Amer. Math. Soc. **68** (1950), 28–46; MR **11**, 369] a identifié les espaces (N), dont la boule unité admet un point extrémal, aux espaces $C(K)$ des fonctions continues sur un espace compact K Stonien. L'objet principal de ce travail est d'étudier les espaces de Banach L qui vérifient la condition suivante: (1) le dual L' de L est un espace (N). L'auteur indique plusieurs conditions équivalentes à (1), dont voici trois: (2) L est isomorphe (avec sa norme) a un espace $L^1(\mu)$, construit sur un espace localement compact muni d'une mesure μ convenable; (3) quel que soit l'espace de Banach E et le sous-espace vectoriel fermé F de E, l'application canonique $L\hat{\otimes}F\to L\hat{\otimes}E$ est un isomorphisme métrique du premier espace dans le second; (4) quel que soit l'espace de Banach E et le sous-espace vectoriel fermé F de E, toute application linéaire continue u de L dans E/F se „relève" en une application linéaire de même norme de L dans E". L'auteur démontre encore le théorème suivant: Soit $C=C_0(K)$ l'espace des fonctions continues, nulles à l'infini, sur un espace localement compact K; si C est isomorphe au dual d'un espace de Banach L, alors K est un espace compact Stonien, C est isomorphe à une algèbre de von Neumann et L est l'ensemble des mesures „normales" sur K. Enfin, l'auteur esquisse une étude des espaces de Banach C qui vérifient la condition suivante: (1') Quels que soient l'espace de Banach E et le sous-espace vectoriel fermé F, l'application canonique $C\check{\otimes}E\to C\check{\otimes}E/F$ est un homomorphisme métrique du premier sur le second; ici le signe $\vee$ remplace le signe $\hat{\otimes}$ employé par l'auteur dans sa Thèse [Mem. Amer. Math. Soc. no. **16** (1955); MR **17**, 763]. Une condition équivalente est: (2') Le dual C' est isomorphe à un espace $L^1(\mu)$. D'autres conditions équivalentes sont indiquées.

J. Sebastião e Silva (Lisbonne).

James, Robert C. Projections in the space (m). Proc. Amer. Math. Soc. **6** (1955), 899–902.

The author proves two theorems concerning projections on the space (m). Theorem 1: If M is a separable subspace of (m) and P is a projection of (m) onto M, then the norm of P satisfies $\|P\|>1$. The proof is based on this argument: If $\|P\|=1$, then M has the Hahn-Banach extension property and it follows that M is the space of all continuous functions on an extremally disconnected compact Hausdorff space. If the latter contains more than a finite number of points, M contains a subspace isometric with (m) and hence is not separable. Theorem 2: If M is a

separable Banach space which is complemented in (m) and if M has an unconditional basis, then M is reflexive. The proof is based on a previous result of the author [Ann. of Math. (2) **52** (1950), 518–527; MR **12**, 616] giving two conditions equivalent to reflexivity for spaces with such bases. The assumption that the second condition is not satisfied leads to the conclusion that l_1 is complemented in (m), a proposition known to be false. *E. R. Lorch.*

Ehrenpreis, Leon. Appendix to the paper "Mean periodic functions I". Amer. J. Math. **77** (1955), 731–733.

Dans un travail précédent [même J. **77** (1950), 293–328; MR **16**, 1122] l'auteur a démontré que, si V est une „variété" de l'espace $\mathcal{A}$ des fonctions entières sur C^n, telle que son ensemble orthogonal V' soit un idéal principal, alors toute fonction $f \in V$ est limite de combinaisons linéaires de monômes exponentials dans V. Maintenant, il démontre que, réciproquement, la thèse n'est pas vérifiée, si V' n'est pas principal. *J. Sebastião e Silva.*

Meschkowski, Herbert. Über beschränkte lineare Transformationen in Banachschen Räumen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **214** (1955), 14 pp.

A study of special analytic functions of bounded linear operators in Banach spaces. The study is based on the calculus of residues developed by Riesz, Lorch, and Dunford. It is proved: If T has no spectrum on the imaginary axis nor on the lines $\Re(z)=\pm\Im(z)$, then $(T^2)^{\frac{1}{2}}=P_1T-P_2T$ where P_1 (P_2) is the projection associated with the spectrum of T lying in the half-plane $\Re(z)>0$ ($\Re(z)<0$). Note that $(T^2)^{\frac{1}{2}}=-(2\pi i)^{-1}\int z^{\frac{1}{2}}(T^2-zI)dz$, where the integration is evaluated around the entire spectrum of T^2. Corollaries are proved to the well-known theorem on the spectral radius of T. Thus, the spectrum of T lies on a closed simple curve C if and only if $\lim\|u(T)^n\|^{\frac{1}{n}}=\lim\|u(T)^{-n}\|^{1/n}=1$, where $u(z)$ is a function which maps conformally the interior of C on the interior of the unit circle. There is also a proof of a statement concerning isolated points in the spectrum which follows from a result of G. Šilov on generalized divisors of zero [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **29** (1940), 83–84; MR **2**, 314]. *E. R. Lorch* (New York, N.Y.).

Lorch, Edgar R. L'integrazione ed i funzionali lineari. Rend. Sem. Mat. Fis. Milano **25** (1953-54), 113–120 (1955).

In this largely expository paper, the author reviews the close connection between integrals and linear functionals, and is led [following Daniell, Ann. of Math. (2) **19** (1918), 279–294] to consider the following situation: R is a set, B a Banach lattice (under the "sup" norm) of bounded real functions on R, and B^* the adjoint of B. A linear functional $F \in B^*$ is called a positive integral if it is a positive functional and is continuous under monotone convergence; an integral is the difference of two positive integrals. The principal result, whose proof will appear elsewhere, asserts that if M is the set (linear subspace) of all integrals in B^*, then there is a unique continuous projection P of B^* onto M such that both P and $1-P$ are positive transformations. This is closely related to a result of Yosida and Hewitt on decomposition of measures [Trans. Amer. Math. Soc. **72** (1952), 46–66; MR **13**, 543]. (Misprints: p. 4, line 26, read "$V_m \cap V_n$" for "$V_n \cap V_n$"; p. 5, line 23, read "$f \geqq 0$" for "$f(x)$", line 32, read "f" for "$f-$".) *V. L. Klee* (Los Angeles, Calif.).

Lorch, Edgar R. L'integrazione e gli ideali massimi. Univ. e Politec. Torino. Rend. Sem. Mat. **13** (1954), 33–38.

Suppose R is a set and K is an algebra of bounded complex-valued functions on R which includes all constant functions, is closed under conjugation, and includes f^{-1} whenever $f \in K$ and f^{-1} is bounded. Let S be the maximal ideal space of K, with $R \subset S \subset K^*$ in the usual way. Let P be the projection of K^* as described (for B^*) in the preceding review (whose principal result is stated to hold for complex- as well as for real-function spaces). It is proved that if $X \in S$, then $PX=0$ if and only if there is a sequence U_n of neighborhoods of X in S such that $\bigcap U^n$ misses R; and if no such sequence exists, then $PX=X$. *V. L. Klee, Jr.* (Los Angeles, Calif.).

Kaplan, Samuel. Biorthogonality and integration. Proc. Amer. Math. Soc. **7** (1956), 109–114.

To produce a theory of biorthogonal systems of more than countable cardinality, the author defines a partial order on the finite subsets of a Banach space E: $X_1 \prec X_2$ means $X_1 \subset [X_2]$, (linear closure of X_2). Two sets $\mathcal{X}=\{\xi\}$ and $\mathcal{Y}=\{\eta\}$ in E resp. E' (conjugate space of E) are called biorthogonal if $(\xi, \eta)^2=(\xi, \eta)$, (ξ, η) is not identically 0 in either variable and the $\mathcal{Y}$-orthogonal sets of $\mathcal{X}$ are directed by $\prec$. (X is called $\mathcal{Y}$-orthogonal if no η has value 1 on more than one member of X.) In these terms, the classical criteria for the "basis" property of a biorthogonal system are given their appropriate and analogous forms, and in terms of bases, the forms of linear functionals are determined. *B. Gelbaum* (Minneapolis, Minn.).

Altman, M. A fixed point theorem for completely continuous operators in Banach spaces. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 409–413.

For the Banach space E, let F be completely continuous on $S=\{x \mid \|x\|=1\}$ to E. Let ϕ be a continuous fixed-point-free involution on S and suppose (a) ϕ takes every $S^n \subset S$ into an S^n. Write Tx for $(I-F)(x/S)/\|(I-F)(x/S)\|$. Theorem 1. If (b) $0 \notin (I-F)S$ and (c) $\inf_S \|Tx-T\phi x\|>0$, then the Leray-Schauder degree of $I-F$ is odd. Theorem 2. If F is on the solid sphere $Q=\{x \mid \|x\| \leqq 1\}$ to E and has no fixed points on S and if (b) is satisfied, then F has a fixed point which is an inner point of Q. The Leray-Schauder degree definition involves a finite-dimensional approximation and in view of (a) Theorem 1 is reduced to the formulation for S^n and the result for this finite-dimensional case has been given by Jaworowski [same Bull. **3** (1955), 289–292; MR **17**, 653]. *D. G. Bourgin.*

Gerčinskiĭ, R. Theorems on existence of implicit functions in functional spaces. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 7–10. (Russian)

This paper is concerned with the existence of solutions of an abstract implicit function (*) $f(x, y)=0$, without the use of either differentiability or topological degree. The main hypotheses required are that the mapping be open and/or closed. Nine theorems are stated without proof, two of which are for general metric spaces, but they cannot be given briefly. As samples, we restate in somewhat simplified form two of the results. It is assumed that H is a complete metric space and that W is an open subset of an infinite-dimensional F-space [see S. Banach, Théorie des opérations linéaires, Warsaw, 1932]. Theorem 6. Let $f: W \times H \to H$ be such that (a) for every $y \in H$ the mapping $x \to f(x, y)$ is open and closed, (b) f is uniformly continuous in both variables (or satisfies a Lipschitz condition in y with constant independent of x), and (c) for $y=y_0$ there are solutions of (*). Then for some $k>0$, if $\operatorname{dist}(y, y_0) \leqq k$ then there exists a solution of (*). Theorem 9. Let $\varepsilon>0$ and f: $S(W, \varepsilon) \times H \to H$ be such that

(a) the mapping $x \to f(x, y)$ sends every subset of W which is open (resp. closed) in $S(W, \varepsilon)$ into an open (resp. closed) set, (b) as above, and (c) if for $y \in H$ there is a solution x of (*), then $x \in W$. Then the set of all y for which (*) has solutions is a component of the space H. This paper seems to be closely related to V. V. Nemyckiĭ [Mat. Sb. N.S. **33(75)** (1953), 545–558; **35(77)** (1954), 174; MR **15**, 719]. There are a number of misprints.

R. G. Bartle (Urbana, Ill.).

Gerčinskiĭ, R. Some sufficient conditions for openness of mappings in functional spaces. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 201–202. (Russian)

It is desired to find conditions sufficient to insure that a mapping between Banach spaces is open. Five theorems are stated without proof, a variety of additional hypotheses on the mapping being employed. Since the most interesting ones are complicated, we shall not state them here. (It may be remarked that the author's Theorem 4 is a special case of a theorem due to L. M. Graves [Duke Math. J. **17** (1950), 111–114; MR **11**, 729], which has been generalized by the reviewer [Math. Z. **62** (1955), 335–346; MR **17**, 62]. In addition, it might be pointed out that the author's use of the term "topological" map, which is not explained, is not synonymous with the term "homeomorphism", since it does not imply that the map be open.)

R. G. Bartle (Urbana, Ill.).

Borisovič, Yu. G. On the influence of error on the convergence of Newton's process for nonlinear functional operations. Kazan. Gos. Univ. Uč. Zap. **113**, no. 10 (1953), 189–192. (Russian)

Let X and V be Banach spaces and let P be a function on X to V which has a bounded second Fréchet derivative on a neighborhood of the desired solution x^* of the equation $P(x)=0$. The author is concerned with the possibility of replacing the modified Newton process

$$x_{n+1}=x_n-\Gamma_0 P(x_n) \quad (n=0, 1, 2, \cdots),$$

where $\Gamma_0=[P'(x_0)]^{-1}$, by allowing for an error at each stage. Thus he considers an iteration

$$(*) \qquad x_{n+1}'=x_n'-\bar{\Gamma}_n P(x'_n) \quad (n=0, 1, 2, \cdots),$$

where $\bar{\Gamma}_n=\Gamma_0+A_{n+1}$. [Actually, the only time the process is explicitly written it appears as $x_{n+1}'=x_n'-\bar{\Gamma}_n P(x_n)$, but judging from the calculations that are later made this must be a misprint.] Theorem 1, which is stated without proof, asserts the convergence of this method to x^*, but as no restrictions are imposed on the $\bar{\Gamma}_n$ this theorem is not to be taken seriously, as stated. Theorem 2 appears to give the hypotheses requisite for Theorem 1 and to establish that

$$\|x_n'-x^*\| \leq q^{n-1}\{\|x_1'-x^*\|+2\|A_1 P(x_0)\|\},$$

where $0<q<1$. [For references on Newton's method see following review.] *R. G. Bartle* (Urbana, Ill.).

Mysovskih, I. P. On convergence of L. V. Kantorovič's method for solution of nonlinear functional equations and its applications. Vestnik Leningrad. Univ. **1953**, no. 11, 25–48. (Russian)

This paper is devoted to the application of Newton's method of solving functional equations. Specifically, if P is a function from one Banach space to another, the equation $P(x)=0$ is to be solved by the iterative procedure

$$x_{n+1}=x_n-[P'(x_n)]^{-1}P(x_n) \quad (n=1, 2, \cdots),$$

or by the modified process

$$x_{n+1}=x_n-[P'(x_0)]^{-1}P(x_n) \quad (n=1, 2, \cdots),$$

where x_0 is sufficiently near the desired solution. The extension of these methods to Banach spaces was made by L. V. Kantorovič [Dokl. Akad. Nauk SSSR (N.S.) **59** (1948), 1237–1240; Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; especially Trudy Mat. Inst. Steklov. **28** (1949), 104–144; Dokl. Akad. Nauk SSSR **76** (1951), 17–20; MR **9**, 537; **10**, 380; **12**, 419, 835]. This paper is an expansion of results announced earlier [ibid. **70** (1950), 565–568; MR **11**, 601] which was reviewed in some detail, and is very close to Trudy Mat. Inst. Steklov. **28** (1949), 145–147 [MR **12**, 419]. The basic difference in the criteria for convergence given by Kantorovič and Mysovskih is that the latter assumes a common bound for the norms of the inverse operators $[P'(x)]^{-1}$ on some neighborhood of the initial approximation x_0, instead of at that point alone. This simplifies the proofs considerably. In this paper the author presents a revised version of the theorem in the Trudy paper and a corresponding uniqueness result. He also establishes convergence for both the regular and the modified processes when a solution is known to lie in a specified neighborhood of x_0. In all four results boundedness of the first and second Fréchet derivatives near x_0 is assumed.

The last half of the paper deals with applications. First the solution of a system of n algebraic equations in n unknowns is treated, the results being related to the work of A. Ostrowski [Comment. Math. Helv. **9** (1936/37), 79–103; Mat. Sb. N.S. **3(45)** (1938), 253–258]. Next, applications are made to non-linear integral equations of the Hammerstein type

$$x(s)=\int_0^1 K(s, t)f(t, x(t))dt.$$

Specific results are obtained but they are too complicated to be stated here. Finally, a boundary-value problem based on $\Delta U=e^U$ in the plane is discussed briefly by reducing it to a Hammerstein integral equation.

Other references on Newton's method in abstract spaces: M. Altman, Bull. Acad. Polon. Sci. Cl. III **3** (1955), 189–193 [MR **17**, 176], R. G. Bartle, Proc. Amer. Math. Soc. **6** (1955), 827–831 [MR **17**, 176], I. Fenyő, Mat. Lapok **3** (1952), 11–46; Acta Math. Acad. Sci. Hungar. **5** (1954), 86–93 [MR **17**, 64; **15**, 964], Y. Hirasawa, Comment. Math. Univ. St. Paul. **3** (1954), 15–27 [MR **16**, 487], and M. L. Stein, Proc. Amer. Math. Soc. **3** (1952), 858–863 [MR **14**, 1094], and the paper reviewed above.

R. G. Bartle (Urbana, Ill.).

See also: Ionescu Tulcea, p. 832; Takehaski, p. 874.

Hilbert Space

Soysal, Selma. Sur certains ensembles ordonnés de projecteurs dans l'espace de Hilbert. I, II. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **18** (1953), 305–321, 323–351. (Turkish summary)

The author considers a strongly closed, commutative family A of orthogonal projections in separable Hilbert space H, totally ordered in the usual order of projections. ($P_1 \leq P_2$ means $P_1P_2=P_2P_1=P_1$.) Corresponding to each vector f, she then defines a vector-valued measure μ_f by assigning the P_2f-P_1f to the subset $\{P|P_2 \geqq P > P_1\}$ of A, and then defines the integral $\int_A \phi(P)dPf$ of a numerical-

valued function of projections. The standard properties belonging to Hilbert-space integrals relative to projection-valued measures are then established in this novel situation. In particular, $\|\int \phi(P)dPf\|^2=\int|\phi(P)|^2d\|Pf\|^2$, so that the set of elements $\int \phi(P)dPf$ form a closed subspace $M(f)$ of Hilbert space. If each of the vector-valued measures μ_g is absolutely continuous relative to μ_f, $M(f)$ is called hypermaximal. It is shown that H can be decomposed as a direct sum $\sum_{i=1}^{\infty}\oplus M(f_i)$ $M(f_n)$ being hypermaximal in the orthocomplement of $\sum_{i=1}^{n-1}\oplus M(f_i)$, and that this decomposition is essentially unique. From the decomposition, a complete set of invariants of A is derived.

J. Schwartz (New York, N.Y.).

Mihlin, S. G. On Ritz's method. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 391–394. (Russian)

Si X est un opérateur non continu dans un espace de Hilbert H on désigne par $D(X)$ l'ensemble de définition de X. Soient A et B deux opérateurs auto-adjoints >0, avec $D(A)=D(B)$, $D(A^{\frac{1}{2}})=D(B^{\frac{1}{2}})$. L'équation $Au=f$, f donné dans H, admet une solution unique. On approche u par u_n défini comme suit; on suppose que B admet un spectre discret; soient λ_n les valeurs propres, φ_n le système orthonormal complet dans H des vecteurs propres. Soit $u_n=\sum_{j=1}^{n}a_j\varphi_j$, les a_j étant des constantes déterminées par la méthode de Ritz:

$$\sum_j a_j(A\varphi_j, \varphi_k)=(f, \varphi_k),\ k=1, \cdots, n.$$

On sait que $u_n\to u$ dans H, $A^{\frac{1}{2}}u_n\to A^{\frac{1}{2}}u$ dans H.

Théorème: $Au_n\to Au$ dans H.

Application: A est un opérateur symétrique elliptique du 2ème ordre donné dans Ω ouvert borné de R^n, de frontière régulière; $D(A^{\frac{1}{2}})$ est l'espace des fonctions $\in L^2(\Omega)$ ainsi que leurs dérivées du premier ordre, et nulles en moyenne sur la frontière (Problème de Dirichlet). Soit $B=-\Delta$, Δ=Laplacien. D'après Mihlin [mêmes Dokl. (N.S.) **78** (1951), 443–446; MR **13**, 16] et Ladyženskaya [ibid. **85** (1952), 481–484; MR **14**, 474] on a $D(A)=D(B)$, on est donc dans les conditions d'application du théorème. Rev.: ceci vaut pour A elliptique d'ordre $2m$, symétrique, $B=-\Delta^m$, d'après O. V. Guseva [ibid. **102** (1955), 1069–1072; MR **17**, 161] et L. Nirenberg [Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742]. *J. L. Lions.*

Vaĭnberg, M. M. On eigenelements of odd potential operators in Hilbert space. Mat. Sb. N.S. **38 (80)** (1956), 3–22. (Russian)

Proofs are given for results stated in previous papers [Dokl. Akad. Nauk SSSR (N.S.) **75** (1950), 609–612; **80** (1951), 309–312; **85** (1952), 261–263; MR **12**, 713; **13**, 353; **14**, 290.] *J. Cronin* (Southbridge, Mass.).

★ **Wermer, J. Report on subnormal operators.** Report of an international conference on operator theory and group representations, Arden House, Harriman, N. Y., 1955, pp. 1–3. Publ. 387, National Academy of Sciences-National Research Council, Washington, D. C., 1955.

Expository paper.

Segal, I. E. Tensor algebras over Hilbert spaces. I. Trans. Amer. Math. Soc. **81** (1956), 106–134.

The author, like many others, is led by the study of quantum field theory to develop here a theory of integration over real Hilbert space. In this paper the author first discusses the algebra A of symmetric tensors over X, a complex Hilbert space. Letting X' be a real Hilbert space and X its complexification, the author then develops his theory of integration over X', obtains an analogue of the Fourier-Plancherel transform, shows that A is unitarily equivalent to the space of "square integrable functions" defined by this integration theory, and considers some other aspects of harmonic analysis for this integration theory.

After boiling away the excess abstraction, the author's integration theory is as follows. For fixed $c>0$ and every k-dimensional subspace $\mathfrak{M}$ of X' define the measure $\mu_{\mathfrak{M}}$ over Borel subsets S of $\mathfrak{M}$ by

$$\mu_{\mathfrak{M}}(S)=(2\pi c)^{-k/2}\int_S \exp\,(-(2c)^{-1}\|y\|^2)d\gamma_{\mathfrak{M}}(y),$$

where $\gamma_{\mathfrak{M}}$ is ordinary k-dimensional Lebesgue measure on $\mathfrak{M}$ identified with euclidean k space. A complex valued function f over $x\in X'$ is said to be tame (and based on $\mathfrak{M}$) if $f(x)=\tilde{f}_{\mathfrak{M}}(x)$ for some finite-dimensional subspace $\mathfrak{M}$, where $\tilde{f}$ is a Borel function on $\mathfrak{M}$ and $x_{\mathfrak{M}}$ is the orthogonal projection of x into $\mathfrak{M}$. For $f\in\mathcal{T}$, the set of bounded tame functions on X', define

$$E(f)=\int_{\mathfrak{M}'}\tilde{f}(y)d\mu_{\mathfrak{M}'}(y),$$

$E(f)$ being independent of $\mathfrak{M}'\supseteq\mathfrak{M}$ on which f is based. $(\mathcal{T}, E)$ forms a complex singular probability algebra which determines a unique family of equivalent probability measure spaces (Ω, P), for each of which $\mathcal{T}$ reduced mod E is one-to-one with a weakly dense set of equivalence classes of bounded, complex valued, P-measurable functions f^+ over Ω and $E(f)=\int_\Omega f^+(w)dP(w)$. The "square integrable functions" on X' are arbitrarily defined as $L_2(\Omega, P)$, although admitedly a P-measure-zero equivalence class of P-measurable functions on Ω generally does not correspond either to a strict function on X' or a class of such. Other theories of integration, connected with X', and the resulting harmonic and vector analysis on X' are given by the reviewer [Pacific J. Math. **2** (1953), 531–553; **5** (1955), 649–662; MR **14**, 963; **17**, 388], P. Lévy [Problèmes concrets d'analyse fonctionnelle [Gauthier-Villars, Paris, 1951; MR **12**, 834] Cameron and Martin; and others. *F. H. Brownell.*

See also: Ionescu Tulcea, p. 832; Singer, p. 874.

TOPOLOGY

Iséki, Kiyoshi. Notes on topological spaces. I. A theorem on uniform spaces. Proc. Japan Acad. **32** (1956), 27–28.

Some elementary results on G_δ subsets of uniform spaces. For instance: A G_δ subset of a complete uniform space must be the intersection of a closed set and a G_δ set in any other uniform space of which it is uniformly a subset. *E. Michael* (Seattle, Wash.).

Ruprecht, Eberhard. Über die Charakterisierung normaler und vollständig regulärer topologischer Boole-Verbände mit Hilfe quasi-stetiger Ortsfunktionen. Math. Nachr. **13** (1955), 289–308.

In order to generalize the characterization of the structure of normal or completely regular topological spaces E by means of continuous real functions over E, the author replaces these topological spaces by topological Boolean algebras and, instead of point functions, he uses Cara-

théodory's "position functions" ("Ortsfunktionen") [C. Carathéodory, S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1938, 27–69; Mass und Integral und ihre Algebraisierung, Birkhäuser, Basel-Stuttgart, 1956, while simultaneously he generalizes continuity to "quasi-continuity". Throughout the notations of G. Nöbeling's Grundlagen der analytischen Topologie [Springer, Berlin-Göttingen-Heidelberg, 1954; MR 16, 844] are employed.

First the author discusses the interpretation of position functions as point functions over a space or as homomorphisms of a Boolean algebra. Then he introduces the notion of "quasi-continuity": Let f be a position function defined on a closed soma $Y>0$ of a topological Boolean algebra; then the author calls f "quasi-continuous" on Y if for every soma X with $0<X\leqq Y$ the relations $\underline{f}(X)\leqq \underline{f}(\bar{X})$ and $\bar{f}(X)\geqq \bar{f}(\bar{X})$ are satisfied, where $\bar{X}$ is the closure of X, while $\underline{f}(X)$ and $\bar{f}(X)$ mean the lower and the upper limit of f on X. If herein $\underline{f}(X)=f(\bar{X})$ and $\bar{f}(X)=f(\bar{X})$, then f is continuous on Y. The conditions are discussed under which a composite position function of quasi-continuous or continuous position functions is again quasi-continuous or continuous.

A topological Boolean algebra $\mathfrak{B}$ is normal if for every pair of closed somas A and B with $A\wedge B=0$ there exist two open somas V and W in $\mathfrak{B}$ with $V\wedge W=0$ and $A\leqq V$, $B\leqq W$. If to every dyadic rational number $t=m/2^n$ ($m=0, 1, 2, \cdots, 2^n$; $n=0, 1, 2, \cdots$) there corresponds an open soma $W_t\in\mathfrak{B}$ such that $t'<t''$ implies $\bar{W}_{t'}\leqq W_{t''}$, then the system (W_t) of these somas W_t is called a "dyadic scale". A topological Boolean algebra $\mathfrak{B}$ is called completely regular if for every soma $A>0$ and every closed soma B of $\mathfrak{B}$ with $A\wedge B=0$ there exists a dyadic scale (W_t) such that $A\wedge W_0>0$ and $B\wedge W_1=0$.

Then the author proves the following main results: (1) In order that the topological Boolean algebra $\mathfrak{S}$ be normal, it is necessary and sufficient that for every pair of closed somas A, B of $\mathfrak{B}$ with $A\wedge B=0$ and to every pair of real numbers a, b with $a\leqq b$, there exists a quasi-continuous position function f with $a\leqq f\leqq b$, which takes on the value a on A and the value b on B. (2) In order that $\mathfrak{B}$ be completely regular, it is necessary and sufficient that for every soma $A>0$, for every closed soma B of $\mathfrak{B}$ with $A\wedge B=0$, and for every pair of real numbers a, b with $a\leqq b$, there exists a quasi-continuous position function f with $a\leqq f\leqq b$, which takes on the value a on a soma X with $0<X<A$ and the value b on B. A simple example shows that in both these theorems quasi-continuity cannot be replaced by continuity. *A. Rosenthal* (Lafayette, Ind.).

Ganea, Tudor. Revêtements et multicohérence; application à un problème de Borsuk. C. R. Acad. Sci. Paris **242** (1956), 725–728.

Let $(\tilde{X}, f)$ be a regular covering space of a connected, locally connected space X, and let $G(\tilde{X}, f)$ be its automorphism group (consisting of homeomorphisms ξ of $\tilde{X}$ onto $\tilde{X}$ for which $f\xi=f$). The author shows that if $(\tilde{Y}, g)$ is similarly a regular covering space for Y, and if further X is paracompact and $\tilde{Y}$ has the extension property for paracompact spaces, then each homomorphism of $G(\tilde{X}, f)$ in $G(\tilde{Y}, g)$ can be induced by a continuous mapping of X in Y. This theorem is used to show that, for any paracompact (normal) connected, locally connected space X, the degree of multicoherence $r(X)$ is $\geqq n$ if and only if some $(\tilde{X}, f)$ exists for which $G(\tilde{X}, f)$ is free on n generators. This generalizes the fact [Eilenberg, Fund. Math. **29** (1937), 101–122] that if X is e.g. a Peano space, $r(X)$ is determined by $\pi_1(X)$. The degrees of multicoherence of some mapping spaces can thence be determined; in particular, $r(S_2^{S_1})=1$, which answers a question raised by Borsuk [ibid. **39** (1952), 25–37; MR **15**, 51]. The author has informed the reviewer that the last assertion in the paper is misprinted; it should read $r(X_1^m)=1$. *A. H. Stone* (Manchester).

Borsuk, K. Families of compacta and some theorems on sweeping. Fund. Math. **42** (1955), 240–258.

Suppose that Φ is a function which assigns to each point x of a metric space X a compact non-empty subset $\Phi(x)$ of a metric space Y. If Φ is upper semi-continuous (for every neighborhood V of $\Phi(x)$ there is a neighborhood U of x with $\Phi(x')\subset V$ for all $x'\in U$), then Φ is called a family of compacta over X. The sets $\Phi(x)$ are called the elements of Φ. The following are the main results. (1) If the elements of Φ are acyclic in dimensions $\leqq k$, and if f and g are continuous on X with $f(x), g(x)\in\Phi(x)$ for every $x\in X$, then for every true k-cycle γ in X, $f(\gamma)$ and $g(\gamma)$ are homologous in $\Phi(X)$. (2) Let X be a compact subset of n-space E^n and a_0 a point exterior to X. Suppose Φ is a family over X whose elements are topological $(n-1)$-spheres in E^n with $x\in\Phi(x)$ and a_0 interior to $\Phi(x)$ for all x. Then each bounded component of E^n-X is contained in $\Phi(X)$. (3) Let f be a simplicial involution on a $(n-1)$-dimensional (curvilinear) pseudo-manifold X in E^n. Let Φ be a family of acyclic compacta over X with $x\in\Phi(x)\subset E^n$ and $\Phi(fx)=\Phi(x)$. Then the bounded component of E^n-X is contained in $\Phi(X)$. *E. E. Floyd* (Charlottesville, Va.).

Efremovič, V. A. Almost topological properties. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 3–8. (Russian)

As the author himself states, this article is of a methodological rather than mathematical character, and contains no assertion whose proof is not trivial. The mapping $x\to x'$ of the metric space R on the metric space R' is almost topological if it is uniformly continuous both ways; it is called a Lipschitz mapping if positive numbers e, L exist such that the distance xy and $x'y'$ of corresponding pairs of points satisfy $e<x'y'/xy<L$. It is shown that a Lipschitz mapping preserves order of contact of curves, sets of Hausdorff measure 0, and absolute continuity of functions. A non-trivial application of these ideas was recently given by A. S. Švarc [Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 32–34; MR **17**, 781].
H. Busemann (Los Angeles, Calif.).

Yarutkin, N. G. Infinitesimally homotopic classes of uniformly continuous mappings. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 32–36. (Russian)

Let $\varphi_0(x)$ and $\varphi_1(x)$ be two uniformly continuous (abbreviated u.c.) mappings of the metric space R_1 in the metric space R_2. A deformation $\varphi_t(x)$ $(0\leqq t\leqq 1)$ of φ_0 into φ_1 is called u.c. if $\varphi_t(x)$ is u.c. on $R_1\times[0, 1]$. If such a u.c. deformation exists, then φ_0 and φ_1 are called infinitesimally homotopic (abbreviated i.h.). Two u.c. mappings $\varphi_0(x)$, $\varphi_1(x)$ of the real axis in the real axis are i.h. if and only if $|\varphi_0(x)-\varphi_1(x)|$ is bounded. A u.c. mapping φ of the line R_1 on a circle C can be represented as the product of a u.c. mapping of R_1 on a line R_2, called the essential part of φ, and a locally isometric mapping of R_2 on C. Two u.c. mappings of a line on a circle are i.h. if and only if their essential parts can be chosen so that they become i.h. Define a space of non-positive curvature as the reviewer [The geometry of geodesics, Academic Press,

New York, 1955, Ch. V; MR 17, 779]. Two u.c. mappings $\varphi_0(x)$, $\varphi_1(x)$ of a simply connected metric space in a simply connected space of non-positive curvature are i.h., if and only if the distance $\varphi_0(x)\varphi_1(x)$ is bounded. Two u.c. mappings φ_0, φ_1 of a metric space R on a compact Riemannian manifold Q are i.h. if and only if they can be written as products of two i.h. mappings of R on the universal covering space $\overline{Q}$ of Q and the same locally isometric mapping of $\overline{Q}$ on Q. *H. Busemann.*

See also: Kapuano, p. 874.

Algebraic Topology

★ **Aleksandrov, P. S. Combinatorial topology. Vol. 1.** Graylock Press, Rochester, N. Y., 1956. xvi+225 pp. $4.95.

Translation of the author's Kombinatornaya topologiya, parts I and II plus Appendix 2 [Gostehizdat, Moscow, 1947; MR **10**, 55]. The Bibliography has been revised and includes more recent material.

★ **Springer, George. Einführung in die Topologie.** Aschendorffsche Verlagsbuchhandlung, Münster, 1955. vii+192 pp. DM 15.00.

This booklet is essentially a printed set of notes of a course of lectures on topology given by the author in 1954–55 while he was Visiting Professor at the University of Münster. The contents are indicated by the chapter headings, which read as follows: I. On topological spaces. II. The theory of complexes. III. Homology groups. IV. Invariance of homology and cohomology groups. V. Manifolds. VI. Topology of compact surfaces. VII. The fundamental group. VIII. Covering manifolds.

For the expert, this booklet is of little use, because all the material may be found in standard texts or well-known papers. It may be of some value to a beginning graduate student who wants to study algebraic topology. *W. S. Massey* (Providence, R.I.).

Barratt, M. G.; and Whitehead, J. H. C. On the second non-vanishing homotopy groups of pairs and triads. Proc. London Math. Soc. (3) **5** (1955), 392–406.

Let Y be an $(r-1)$-connected CW-complex ($r\geqq 3$), not necessarily finite. The authors study $\mathrm{Ext}(Q, G)$ and apply it to the computation of $\pi_{r+1}(Y)$. Then, they calculate $\pi_{r+1}(K, L)$, of an $(r-1)$-connected pair (K, L) with L 2-connected. Assuming A, B to be CW-complexes, respectively $(m-1)$- and $(n-1)$-connected, they analyse the structure of $\pi_{m+n+1}(A\times B, A\vee B)$. This is applied to calculate $\pi_r(X; A, B)$ $(r=m+n)$, where $X=A\cup B$, and A, B with non-vacuous intersection. As illustration they compute these groups in some concrete spaces. *J. Adem* (Mexico, D.F.).

Okamoto, Einosuke. On $(k+1)$-ad homotopy groups. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 93–99.

The author gives a proof of the fact that the homotopy sequence of a $(k+1)$-ad is exact and a couple of simple applications of this fact. [For another recent proof of this result, see Y. Inoue, Math. Japon. **3** (1955), 97–102; MR **17**, 653.] *W. S. Massey* (Providence, R.I.).

Jou, Yuh-lin. Pseudomanifold and manifold homotopy groups. J. Chinese Math. Soc. (N.S.) **1** (1951), 164–206. (Chinese summary)

Let n be a positive integer and M a closed orientable pseudomanifold. By the nth M-homotopy group $\pi_M{}^n(Y)$ of a space Y the author designates the nth homotopy group $\pi^n(Y^M)$ of the function space Y^M based at a constant $y_0{}^M$. Particular cases are the ordinary homotopy groups $\pi^n(Y)$, the Abe groups $\varkappa^r(Y)=\pi^1_{S^{r-1}}(Y)$, the torus homotopy groups $\tau^r(Y)=\pi^1_{T^{r-1}}(Y)$, and Hu's abhomotopy groups $\varkappa^n_r(Y)=\pi_{S^{n-r-1}T^{r+1}}(Y)$. Various properties of the torus homotopy groups are extended to the M-homotopy groups. The most interesting of these are the following. (1) If M is an m-dimensional closed orientable manifold, then $\pi^{n+m}(Y)$ can be isomorphically imbedded in $\pi_M{}^n(Y)$ whenever the product $M\times E^n$ of M and an n-cell E^n can be topologically imbedded in Euclidean $(n+m)$-space; (2) A product

$$\alpha\cdot\beta\in\pi_{K\times H\times L}{}^{m+n+1}(Y)$$

for any $\alpha\in\pi_{K\times H}{}^{m+1}(Y)$ and $\beta\in\pi_{K\times L}{}^{n+1}(Y)$ is defined which generalizes the Whitehead product for three arbitrary closed orientable pseudomanifolds K, H, L. The isomorphisms

$$\varphi:\pi_{K\times H}{}^{m+1}(Y)\to\pi^1{}_{S^m\times S^n\times K\times H\times L}(Y),$$

$$\varphi':\pi_{K\times L}{}^{n+1}(Y)\to\pi^1{}_{S^m\times S^n\times K\times H\times L}(Y),$$

$$\overline{\varphi}:\pi_{K\times H\times L}{}^{m+1}(Y)\to\pi^1{}_{S^m\times S^n\times K\times H\times L}(Y),$$

can be so chosen that

$$\overline{\varphi}(\alpha\cdot\beta)=\varphi(\alpha)\varphi'(\beta)\varphi(\alpha^{-1})\varphi'(\beta^{-1}).$$

Relative and triad M-homotopy groups are also considered; they offer no surprises. *R. H. Fox.*

Ganea, Tudor. Catégorie 1-dimensionnelle et homomorphismes de groupes fondamentaux. C. R. Acad. Sci. Paris **242** (1956), 1407–1410.

It is known that if X is a polyhedron and Y is aspherical in dimensions 2, 3, $\cdots$, $\dim X-1$ then every homomorphism of $\pi_1(X)$ into $\pi_1(Y)$ is induced by a continuous mapping of X into Y. The author shows that the same conclusion may be drawn if Y is only required to be aspherical in dimensions 2, 3, $\cdots$, $\mathrm{cat}_1 X-2$ where $\mathrm{cat}_1 X$ is the 1-dimensional category previously defined by the author [Acad. R. P. Române. Bull. Şti. Secţ. Sti. Mat. Fiz. **5** (1953), 127–133; MR **17**, 393]. For this statement the integer $\mathrm{cat}_1 X$ is shown to be best possible. *R. H. Fox* (Princeton, N.J.).

Weier, Josef. Ein Approximationssatz. Compositio Math. **12** (1956), 185–216.

The author continues his study of homotopic deformations, with special attention to fixed points. If $F:X\times I\to X$ is a deformation of X then $F|X\times t$, $0\leqq t\leqq 1$, shall be denoted F^t. The main result is the Approximation Theorem: Let F be a deformation of a compact topological manifold P with metric ϱ, and ε be a positive number. Then there is an integer ζ and a deformation G of P such that $\varrho(F^t, G^t)<\varepsilon$, $0\leqq t\leqq 1$, and having the properties: (1) for $0\leqq t\leqq 1$, G^t has at most ζ fixed points, and (2) if F^0 and F^1 have a finite number of fixed points, then G can be chosen so $G^0=F^0$, $G^1=F^1$. *S. Stein.*

★ **Hirsch, Guy. La structure homologique des espaces fibrés et leur classification.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 59–63. Fédération belge des Sociétés Scientifiques, Bruxelles.

An expository paper on fiber spaces where the author discusses several problems. *J. Adem* (Mexico, D.F.).

Boltyanskiĭ, V. G. Second obstructions for cross-sections. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 99–136. (Russian)

This paper constitutes a detailed account, with proofs, of the results announced in an earlier paper [Dokl. Akad. Nauk SSSR (N.S.) **85** (1952), 17–20; MR **14**, 307]. In addition there is a useful introductory section on the general theory of fibre-bundles, together with a treatment of the first obstruction. *P. Hilton* (Manchester).

Boltyanskiĭ, V. Infinite-dimensional homologies and cohomologies. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1141–1143. (Russian)

Let H be Hilbert space (with metric ϱ) and let a subset M and real numbers r_0, $r_1>0$ be given such that (i) $\varrho(x, y)\geqq r_0$ for all distinct $x, y \in M$, (ii) $\varrho(x, M)\leqq r_1$ for all $x \in H$. A point $x \in H$ is said to be of finite order (with respect to M) if there exists $\varepsilon>0$ such that only a finite number of points $m \in M$ satisfy the inequality $\varrho(x, m)<\varrho(x, M)+\varepsilon$. If x is of finite order, $\mu(x)$, the 'zone' of x, is defined to be the (finite) set of points $m \in M$ such that $\varrho(x, m)=\varrho(x, M)$. [Reviewers note: the author calls $\mu(x)$ the 'attractive set' of x.] For a given finite subset, μ, of M, $\tau(\mu)$ is defined to be the set of points x of H whose zone is μ. Then the non-empty sets $\tau(\mu)$ are called the cells of a cellular subdivision of H. Certainly every point x of finite order belongs to precisely one cell, namely $\tau(\mu(x))$, and the points of finite order form an open dense subset of H. The dimension of the linear hull $L(\mu)$ of μ is called the defect of $\tau(\mu)$. If μ consists of the points $m_1, \cdots, m_k$, of M, the equations $\varrho(y, m_1)=\cdots=\varrho(y, m_k)$, $y \in H$, determine a linear space K, the orthogonal complement of $L(\mu)$. Then K contains $\tau(\mu)$ and is called the carrier of $\tau(\mu)$.

If $\tau=\tau(\mu)$, $\tau'=\tau(\mu')$, $\mu\neq\mu'$, then τ' intersects $\bar{\tau}$ if and only if $\mu\subset\mu'$; in that case $\tau'\subset\bar{\tau}$ and τ' is said to lie in the boundary of τ. The defect of τ' is then greater than that of τ. For a given τ, there are infinitely many τ' with $\tau'\subset\bar{\tau}$ (indeed, infinitely many with defect greater by any given positive integer n), but, for a given τ', there are only initely many τ with $\bar{\tau}'\subset\tau$. If defect $\tau=d$, defect $\tau'=d+2$, there are precisely two cells of defect $d+1$ in the boundary of τ and having τ' in their boundary. It is possible to introduce a notion of orientation into the cellular subdivision and thence of incidence numbers, $[\tau:\tau']$, between oriented cells in such a way that

$$\sum_{\tau}[\tau_1:\tau][\tau:\tau_2]=0,$$

where τ_1, τ_2 are oriented cells of defect r, $r+2$ and τ ranges over all cells of defect $r+1$. Thus, for any subcomplex P of the given cellular subdivision, homology groups and cohomology groups may be defined. The chain group of defect r, $C_{\infty-r}(P, G)$ is the group of functions of the oriented cells of defect r with values in G and the boundary maps $C_{\infty-r}$ into $C_{\infty-r-1}$. The cochain group of defect r, $C^{\infty-r}(P, G,)$is the group of formal finite sums of oriented cells of defect r, with coefficients in G, and the coboundary maps $C^{\infty-r}$ into $C^{\infty-r+1}$. The resulting homology and cohomology groups are independent, up to isomorphism, of the choice of orientation.

P. J. Hilton (Manchester).

Wu, Wen-Tsün. On the realization of complexes in euclidean spaces. I. Acta Math. Sinica **5** (1955), 505–552. (Chinese. English summary)

In this paper the author gives a generalization of a topological invariant introduced by van Kampen [Abh. Math. Sem. Hamburg. Univ. **9** (1932), 72–78] in studying the realization problem of an n-dimensional complex in the $2n$-dimensional euclidean space.

Let K be a n-dimensional finite simplicial complex and denote by $\tilde{K}^*$ the subcomplex of the product complex $K\times K$ consisting of all cells $\sigma\times\tau$ such that σ and τ are simplexes in K having no vertex in common. Identifying each pair $\sigma\times\tau$ and $\tau\times\sigma$ of $\tilde{K}^*$ to a same cell $\sigma*\tau=\tau*\sigma$, one obtains a cell complex K^*. Suppose that the cells $\sigma \in K$ are oriented and let us orient the cell $\sigma*\tau$ of K^* as $\sigma\times\tau$ in $K\times K$ and

$$\tau*\sigma=(-1)^{\dim\sigma\dim\tau}\sigma*\tau.$$

Let K_1 be an arbitrary simplicial subdivision of K and map K_1 linearly into the euclidean m-space R^m such that any two simplexes of K_1 are in general position whenever possible. Then, with respect to a fixed orientation of R^m, to any two disjoint simplexes σ, τ in K with sum of dimensions just equal to m there corresponds a definite intersection number $\phi(\sigma', \tau')$ where σ', τ' denote the images of σ, τ in R^m under the linear map described above. Let I_m denote the group of integers if m is even and the group of integers mod 2 if m is odd, and let ϱ_m denote the corresponding identity or reduction mod 2. Then an m-dimensional cochain $\phi \in C^m(K^*, I_m)$ may be defined by

$$\phi(\sigma*\tau)=(-1)^{\dim\sigma}\varrho_m\phi(\sigma', \tau').$$

This cochain ϕ is a cocycle and the cohomology class Φ^m of ϕ does not depend on the choices of the subdivision K_1 or the linear map $K_1\to R^m$. The classes $\Phi^m \in H^m(K^*, I_m)$, $m=1, 2, \cdots$, are called the imbedding classes of K. If K can be realized in R^m, then $\Phi^m=0$. If m is even, then $2\Phi^m=0$ but in general Φ^m is non-trivial. In case $m=2n, n=\dim K$, then Φ^m reduces to the invariant of van Kampen.

S. T. Hu (Detroit, Mich.).

Adams, J. F. A new proof of a theorem of W. H. Cockcroft. J. London Math. Soc. **30** (1955), 482–488.

The author gives the following extension of some results proved by Cockcroft [Proc. London Math. Soc. (3) **4** (1954), 375–384; MR **16**, 62]. Let L be a connected CW-complex and K a complex obtained by attaching a set of 2-cells to the 1-skeleton of L. Assume $\pi_2(L)\neq 0$ and $\pi_2(K)=0$. Let P be the kernel of the natural map $\pi_1(L)\to\pi_1(K)$. Then P made abelian is free abelian on $\omega\varepsilon$ generators, where ω is the order of $\pi_1(K)$ and ε is the number of 2-cells we attach. Moreover, P has a non-trivial subgroup which is its own commutator subgroup. *J. Adem.*

See also: Ganea, p. 881; Atiyah, p. 894.

GEOMETRY

★**Klein, F. Famous problems of elementary geometry. The duplication of the cube, the trisection of an angle, the quadrature of the circle.** Dover Publications, Inc., New York, 1956. xi+92 pp. Paperbound: $1.00; clothbound: $1.50.

Reprint by photo-offset of the 1930 edition of Beman and Smith's translation [Stechert, New York].

Cavallaro, Vincenzo G. L'ettadecagono regolare. Period. Mat. (4) **33** (1955), 309–312.

Andronov, I. K. On the equicompositeness of the parallelehedra of E. S. Fedorov with equal volume. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 125–132. (Russian)

In 1885 E. S. Fedorov showed in his "Elements of a theory of figures" [cf. MR 15, 923] that there are exactly 5 symmetric convex polyhedra in E^3 with which the E^3 can be parqueted. The present paper shows that (in contrast to the behavior of general convex polyhedra in E^3) any two of these five polyhedra, if they have the same volume, are equicomposite, i.e. can be decomposed into pairwise congruent convex polyhedra. *H. Busemann.*

Andronov, I. K. On approximations in resectioning according to the method of equicompositeness of polyhedra with equal volume. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 133–137. (Russian)

Andronov, I. K. On necessary and sufficient conditions for equicompositeness of equivalent plane figures of zero genus. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 138–144. (Russian)

Let C_i $(i=1, 2)$ be a closed Jordan curve formed by a finite number of straight segments and extruding or intruding convex arcs; a convex arc is intruding (extruding) if every interior point of a chord of the arc lies outside (inside) the closed domain D_i bounded by C_i. If D_1 and D_2 have equal areas, then they can be decomposed into equal numbers of pairwise congruent domains if, and only if, the extruding arcs of C_1 and C_2 and the intruding arcs of C_1 and C_2 are pairwise congruent.

H. Busemann (Los Angeles, Calif.).

Kolobov, P. G. A configuration connected with a heptagon described about a conic section. Rostov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 2, 55–62. (Russian)

When seven lines $a_1, a_2, \cdots, a_7$ are tangent to a conic, and point (a_1a_2) is connected with (a_4a_5) by a line, and similarly point (a_2a_3) with (a_5a_6), then these lines intersect in a point S. Going around the conic section from (a_1a_2) to (a_2a_3) to (a_3a_4) etc. in order, we get 7 points lying on a conic. (This result is ascribed to K. A. Andreev.) Relabeling the 7 lines we obtain 420 points S, each the Brianchon point of a particular circumscribed hexagon. The configuration of these 420 points has interesting properties, e.g. they lie in groups of 10 on conic sections and also in groups of 10 on cubic curves.

D. J. Struik (Cambridge, Mass.).

Gyarmathi, László. Bemerkungen zum ursprünglichen Beweis des Satzes von Pohlke. Acta Univ. Debrecen. **1** (1954), 145–150; additamentum ad **1** (1955), 20. (Hungarian. German summary)

Gasapina, Umberto. Le similitudini piane. Period. Mat. (4) **33** (1955), 276–296.

Backes, F. Sur les sphères à deux paramètres dont les points caractéristiques sont répartis sur un cercle. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 153–162.

Rukavicyn, I. N. On an invariant of a metric bundle of spheres with the radical center at infinity. Irkutsk. Gos. Univ. Trudy **8** (1953), no. 1, 12–17. (Russian)

Such a bundle of spheres is given by the three spheres $K_i(x^2+y^2+z^2)+2L_ix+2M_iy+2N_iz+P_i=0$ $(i=1, 2, 3)$ if a) their centers are not on a line, b) $K_i|\bar{C}_i|\cos\delta_i=$ const, where the $\bar{C}_i$ are the radius vectors of the centers and δ_i is the angle of $\bar{C}_i$ and $\bar{C}_i'$, the $\bar{C}_i'$ being the reciprocal system of the $\bar{C}_i$. The centers of the bundle $A_1\xi+A_2\eta+A_3\zeta+A_4=0$, where the A are real numbers and ξ, η, ζ are proportional to the powers of a point $P(x, y, z)$ with respect to the spheres, are in a plane. The invariant is

$$(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2$$
$$+2(\xi_2-\xi_1)(\eta_2-\eta_1)\cos\angle(\bar{C}_1', \bar{C}_2')$$
$$+2(\eta_2-\eta_1)(\zeta_2-\zeta_1)\cos\angle(\bar{C}_2', \bar{C}_3')$$
$$+2(\zeta_2-\zeta_1)(\xi_2-\xi_1)\cos\angle(\bar{C}_3', \bar{C}_2'),$$

which is unchanged under an alteration of the basic spheres. *D. J. Struik* (Cambridge, Mass.).

Glogovskiĭ, V. V. Equidistant sets. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. **1** (1955), 72–90 (1 insert). (Russian)

A brief discussion of loci in E^n equidistant from two given loci is followed by giving explicitly the loci in E^3 equidistant from any two (including two sets of the same type) of the following loci: point, line, plane, sphere, circular cylinder. This is applied to the solution, from the point of view of descriptive geometry, of the following two problems: to construct a sphere with center on a given line, tangent to another given line and to a given plane; to construct a sphere passing through two given points and tangent to two given planes not containing the points.

H. Busemann (Los Angeles, Calif.).

Buĭmola, G. L. Investigation of primitive errors of geometric constructions. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. **6** (1954), 71–81. (Ukrainian. Russian summary)

A "practical" point, as opposed to an ideal point, is considered as an ideal circle of radius ω_2, and a practical" line as a strip of width $2\omega_1$ bounded by parallel ideal lines. It is stated that, in actual drawing, ω_2 lies between 0.08 and 0.13 mm, and $2\omega_1$ between 0.13 and 0.26 mm. Operating with practical points and lines requires a preliminary discussion, when points are 'practically" indistinguishable or a point and a line are "practically" incident. After this discussion the author evaluates the errors, in terms of ω_1 and ω_2, which occur in the simplest geometric constructions. *H. Busemann* (Los Angeles, Calif.).

Kostovskiĭ, A. N. On the possibility of solving problems in construction by compasses with a bounded spread of the legs. L'vov. Gos. Univ. Uč. Zap. **29**, Ser. Meh.-Mat. no. **6** (1954), 88–97. (Russian)

It is shown that in the Mascheroni constructions in the euclidean plane by means of compass alone it suffices to use a compass which draws circles whose radii are bounded say $\leq R$. As to circles of radii greater than R, it is shown, just as for straight lines, that arbitrarily many of their points can be constructed. *H. Busemann.*

Nahimovskaya, A. Pencils of curves of constant curvature on the Lobačevskiĭ plane. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **19** (1954), 44–55. (Russian)

If u, v are Cartesian coordinates of a hyperbolic plane with space constant k and $x=\cosh k^{-1}u \sinh k^{-1}v$, $y=\sinh k^{-1}u$, $z=\cosh k^{-1}u \cosh k^{-1}v$ (i.e. x, y, z are Weierstrass coordinates), then L: $Ax+By+Cz-D=0$ represents a curve of constant curvature (and all such curves are representable in this form, $D=0$ corresponding to straight lines). If $L'=0$ is a second curve of constant curvature, then $\lambda L+\lambda' L'=0$ is a pencil of such curves. The structure of these pencils is investigated. The main concept is the so called radical axis of the pencil, which for

$D^2+D'^2\neq 0$ is the only straight line in the pencil and has many properties analogous to the well-known ones in case both $L=0$ and $L'=0$ are circles. *H. Busemann.*

Mokriščev, K. K. On solvability with restricted means of construction of problems of second degree in the Lobačevskiĭ plane. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 32 (1955), no. 4, 15–27. (Russian)

It is shown that second-order construction problems in the hyperbolic plane can be solved by using one of the following sets of instruments: 1) compasses drawing circles and limit circles; 2) compasses drawing equidistant curves; 3) a ruler whose edges are parallel (or asymptotic) where a point has been marked on one edge such that the perpendicular to the edge at this point intersects the other edge.

As preliminary steps to 2) it is first shown that compasses drawing circles and equidistant curves, or limit circles and equidistant curves, suffice. *H. Busemann.*

Rogačenko, V. F. On the possibility of solution of problems of construction of the 2nd degree in the Lobačevskiĭ plane by means of compasses and hypercompasses. L'vov. Gos. Univ. Uč. Zap. 22, Ser. Fiz.-Mat. no. 5 (1953), 72–83. (Russian)

A. S. Smogorževskiĭ has shown [Geometric constructions in the Lobačevskiĭ plane, Gostehizdat, Moscow-Leningrad, 1951; MR 14, 575] that all second-order constructions in the hyperbolic plane can be performed by means of compasses drawing circles, limit circles and equidistant curves (analogously to the construction of Mascheroni by compass alone in E^2). It is shown here that it suffices to use compasses drawing circles and equidistant curves. As in the case of E^2 a line is considered constructed, if two of its points have been constructed. A limit circle is considered as constructed if one of its radii and its intersection with the limit circle have been constructed.

H. Busemann (Los Angeles, Calif.).

Samorukov, B. N. On a theorem of Chasles. Rostov. Gos. Ped. Inst. Uč. Zap. 1953, no. 2, 63–69. (Russian)

Purely synthetic proof of Chasles' generalization of Pascal's theorem to a tetrahedron whose edges are intersected in twelve points by a quadric ("Aperçu historique"), a proof which also leads to Bobillier's theorem on a tetrahedon inscribed in a quadric. *D. J. Struik.*

Gulyaeva, L. A. On a mapping of three-dimensional projective space into a system of conic sections in the plane. Kirov. Gos. Ped. Inst. Uč. Zap. 1953, no. 7, 29–47. (Russian)

In a plane P let $\bar{S}_3$ be a system of conics passing through two fixed points M', M'', and Σ a non-degenerate ruled quadric passing through M' and M''. If from a point M of Σ such that the lines MM', MM'' are two generators of Σ, the quadric is projected stereographically upon the plane P, a one-to-one correspondence is established between the conics of the system $\bar{S}_3$ and the conic sections of Σ. According to the author, A. A. Glagolev extended this transformation by considering the poles, for Σ, of the planes of the latter conics [dissertation, Moscow, 1946]. A projective one-to-one correspondence is thus established between the points of the projective three-space S_3 and the system $\bar{S}_3$ including a method of constructing corresponding elements of the two systems. If A is a point in S_3, its polar plane, for Σ, cuts Σ along a conic which is projected from M upon the plane P along a conic of the system $\bar{S}_3$, and conversely. This correspondence translates any figure in the projective three-space S_3 into a relation between some elements of $\bar{S}_3$. The polarity with respect to Σ is of particular interest in this connection.

The following examples picked more or less at random may give an idea of the power and scope of the transformation.

1. Desargues's triangle theorem. If in the system of conics $\bar{S}_3$ one takes two triads of conics A, B, C, and A_1, B_1, C_1 (not belonging to the same pencil) such that the three pencils of conics (A, A_1), (B, B_1), (C, C_1) have a conic in common, then the three conics obtained as the intersections of the three pairs of pencils (A, B) and (A_1, B_1), (B, C) and (B_1, C_1), (C, A) and (C_1, A_1), belong to the same pencil of conics.

2. The totality of conics of $\bar{S}_3$ which osculate a conic of the plane P not belonging to $\bar{S}_3$ but passing through one of the points M', M'', form a one-dimensional system of the third order to which in S_3 corresponds a skew cubic. (The author mentions that such a system of conics was obtained by L. Berzolari in an investigation dealing with differential equations [Boll. Un. Mat. Ital. (2) 2 (1939), 1–10; MR 1, 300], but she gives no other details.)

3. a. To the conics of a net $\bar{S}_2$ of $\bar{S}_3$ correspond in S_3 the points of a plane Q. The conic of $\bar{S}_2$ which corresponds to a point A' common to Q and Σ degenerates into the two lines AM', AM'' which join the point $A=(P, MA')$ to the points M', M''. Hence the locus of the point A (i.e., the Steinerian of the net S_2) is the conic φ into which the conic $\varphi'=(Q, \Sigma)$ is projected from M upon the plane P. The conic φ and the net $\bar{S}_2$ are said to be "conjugate". b. If the nets whose conjugate conics form a pencil p of $\bar{S}_3$, have a pencil q in common, the nets conjugate to the conics of q have the pencil p in common.

The subject is treated synthetically throughout. Bibliographical references are few and those given are fragmentary. Cross references within the paper are scant.

N. A. Court (Norman, Okla.).

Norkin, S. On projective images of regular nets. Vologod. Ped. Inst. Uč. Zap. 13 (1953), 147–152.

The euclidean plane can be covered with regular triangles, squares and regular hexagons. They form regular sets. When projective images are constructed, then an arbitrary nondegenerated quadrangle determines one image of a square net, and an arbitrary triangle can still be made to be part of ∞^1 regular triangular nets. Conditions are given that a hexagon may be mapped into a regular hexagonal net. *D. J. Struik* (Cambridge, Mass.).

Domšlak, Yu. I.; Levinov, A. M.; and Semyanistyĭ, V. I. Numerical invariants and geometrical characteristics of linear complexes and paratactic congruences. Dokl. Akad. Nauk Azerbaĭdžan. SSR. 9 (1953), 71–74. (Russian. Azerbaijani summary)

A theory of "figures of symmetry" has been developed by B. A. Rozenfel'd in several papers [e.g., Dokl. Akad. Nauk SSSR (N.S.) 57 (1947), 543–546; 59 (1948), 1057–1060; Mat. Sb. N.S. 22(64) (1948), 457–492; MR 9, 249; 10, 66]. Every two figures of symmetry determine numerical invariants and geometrical characteristics. This theory is now applied to linear complexes in projective P_n and paratactic congruences of the non-euclidean spaces S_n (ordinary ones), ${}^*S_n$ (indefinite metric), $K_n(i)$ and $K_n(\varepsilon)$ (complex spaces discussed with complex numbers based on either i or ε, $i^2=-1$, $\varepsilon^2=0$). Paratactic congruences of lines satisfy conditions of congruence, so

that they pass into themselves by involutionary motions. The linear complexes are aggregates of $\frac{1}{2}(n-1)$-dimensional planes which pass into each other by null-systems $u_i=a_{ij}x^j$, $a_{ij}=-a_{ji}$. One of the results is that the geometrical characteristics of two such complexes consist of $\frac{1}{2}(n-1)$ pairwise intersecting lines pointwise fixed under the product of the null-systems which determine these complexes. *D. J. Struik* (Cambridge, Mass.).

Longo, Carmelo. Sui fasci di complessi lineari di piani in S_5. Rend. Sem. Mat. Univ. Padova **24** (1955), 300–311.

A contravariant trivector in E_n is but for a scalar factor a linear S_3-complex in S_n ($S_n=n$-dimensional projective space). These complexes were investigated and classified by the author up to $n=7$ in Ann. Mat. Pura Appl. (4) **37** (1954), 61–138 [MR **16**, 627] and for $n=8$ in Rend. Mat. e Appl. (5) **14** (1955), 510–524 [MR **16**, 1147]. Here bundles of ∞^1 S_3-complexes in S_5 are considered. According to Segre [Ann. Mat. Pura Appl. (3) **27** (1918), 75–123] in the general case the bundle contains four special complexes, three singular lines and three invariants. The author gives a canonical form for this general case and deals with the following special cases: 1. Four special complexes exist but there are only one or two invariants. 2. Two special complexes coincide. 3. Only one special complex exists. The bundles can be represented by straight lines in the S_{19} of all S_2 in S_5. A full classification is equivalent to a classification of all tangent lines of the V_{18}^4 representing all special complexes. Canonical forms are given and many interesting geometric properties are proved.

J. A. Schouten (Epe).

Padma, N. Linear tensors and region-complexes. J. Indian Math. Soc. (N.S.) **19** (1955), 73–93.

The author refers to earlier papers [same J. (N.S.) **12** (1948), 107–116; **13** (1949), 91–104; MR **10**, 572; **11**, 397]. In this paper he studies the polar and dual-polar complexes and their singular elements. The general purpose is to express a complex with a complete axis and a focal region as a polar with respect to another complex. This leads to the proof of the quadratic identities given by Weitzenböck [Invariantentheorie, Noordhoff, Groningen, 1923, p. 115–117], which constitute a necessary and sufficient condition for the simplicity of a tensor.

J. A. Schouten (Epe).

Archbold, J. W. Projective geometry over an algebra. Mathematika **2** (1955), 105–115.

The author considers the generalization of projective geometry that is obtained by considering homogeneous coordinates $(x_0, x_1, x_2, \cdots, x_{h+1})$, where x_i ($i=0, 1, \cdots, h+1$) is contained in an algebra A. A particularly detailed analysis when A is taken to be the algebra over GF(2) generated by the matrices $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, is developed. An indication is given as to the application of these results to the construction of group devisible partially balanced incomplete block designs. *L. J. Paige.*

Lombardo-Radice, Lucio. Sui piani finiti a configurazione di Fano universale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 154–161.

Im Anschluss an die von Hanna Neumann [Arch. Math. **6** (1954), 36–40; MR **16**, 739] aufgeworfene Frage nach der Struktur projektiver Ebenen über einer Veblen-Wedderburn-Algebra als Koordinatenbereich, in denen das Theorem 7_3 universell gilt, d.h. in denen alle Vierecke kollineare Nebenecken haben, beweist Verf.: Eine endliche projektive Ebene, in der das Theorem 7_3 universell gilt und in der alle Homologien bezüglich einer festen Achse und festem Centrum in nicht vereinigter Lage existieren, ist eine Desargues'sche Ebene, daher eine projektive Ebene über einem Körper. *R. Moufang.*

Sce, Michele. Sul postulato della continuità della retta ed alcune sue applicazioni alla geometria proiettiva. Period. Mat. (4) **33** (1955), 297–308.

Brown, L. M. A configuration of points and spheres in four-dimensional space. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954–55), 145–149 (1956).

The author has discovered the following four-dimensional analogue for Clifford's chain of points and circles. In Euclidean (or conformal) 4-space, consider seven 3-spheres $S1, S2, \cdots, S7$ through a point $P8$. The first four spheres meet again in a point $P8(567)$; there are 35 such points. Define $S12$ to be the sphere through the five points $P8(123), \cdots, P8(127)$; there are 21 such spheres. By a theorem due to J. H. Grace [Trans. Cambridge Philos. Soc. **16** (1898), 153–190, p. 181], the six spheres $S12, \cdots, S17$ all meet in a point $Q1$; there are 7 such points. Similarly the six spheres $S1, S2, S34, \cdots, S37$ meet in a point $Q3(128)$; there are 105 such points. The four spheres $S1, S56, S57, S67$ meet in $P8(567)$ and a second point $P1(567)$; there are 140 such points. The 20 points $P1(abc)$ and $P2(abc)$, where abc is any triad from 34567, lie on a sphere $T12$; there are 21 such spheres. The six spheres $S12, S13, S14, T56, T57, T67$ meet in a point $Q1(567)$; there are 140 such points. The seven spheres $S1, T12, \cdots, T17$ meet in a point $P1$; there are 7 such points. The five spheres $S12, T34, \cdots, T37$ meet in a point $P3(128)$; there are 105 such points. The 20 points $Q1(abc)$, where abc is any triad from 234567, lie on a sphere $T1$; there are 7 such spheres. The seven spheres $T1, \cdots, T4, T56, T57, T67$ meet in a point $Q8(567)$; there are 35 such points. Finally, the 7 spheres $T1, \cdots, T7$ meet in a point $Q8$. We thus have a configuration of 576 points and 56 spheres, with 7 spheres through each point and 72 points on each sphere. This slightly modified version of the author's notation becomes still more symmetrical when we write $Ta8$ for Sa, and $Sa8$ for Ta ($a=1, \cdots, 7$). Since the incidences are the same as in the seven-dimensional polytope 1_{32}, with its 576 vertices and 56 cells 1_{32} [Coxeter, Philos. Trans. Roy. Soc. London. Ser. A. **229** (1930), 329–425, p. 401], the group of the configuration is $[3^{3,2,1}]$, of order $8 \cdot 9!$. It follows also that the 28 pairs of "opposite" spheres are related like the 28 bitangents of the quartic curve of genus 3. *H. S. M. Coxeter* (Toronto, Ont.).

Sauer, Robert. Wackelige Zwölfkante. Abh. Math. Sem. Univ. Hamburg **20** (1955), 1–9.

A "twelve-edge" consists of twelve straight-line segments [=edges] of fixed lengths. It is either a generalized octahedron or hexahedron. In the first [second] case it has 6 [8] vertices and 4 [3] edges meet at each of them. Its 8 [6] "sides" are plane triangles [plane or skew quadrangles; the latter will lie on some hyperbolic paraboloids].

Let $\mathfrak{x}_i+\varepsilon\mathfrak{x}_i$ denote the vectors from the origin to the vertices of the twelve-edges $(\mathfrak{x}_*)$ [ε=const; $i=1, 2, \cdots, 6$ or $i=1, 2, \cdots, 8$]. Suppose the lengths of corresponding edges of $(\mathfrak{x})=(\mathfrak{x}_0)$ and $(\mathfrak{x}_*)$ differ only by $O(\varepsilon^2)$. Then $(\mathfrak{x})$ is called "flächenstarr wackelig" ["ebenenstarr wackelig"] if corresponding sides [the corresponding "4-edges" or

"3-edges" consisting of the edges meeting at each vertex] are congruent up to $O(\varepsilon^2)$. If a twelve-edge is shaky in both meanings, it is rigid up to $O(\varepsilon^2)$. This case is excluded. The relationship between $(\mathfrak{x})$ and $(\bar{\mathfrak{x}})$ is symmetric. With $(\mathfrak{x})$, $(\bar{\mathfrak{x}})$ is shaky and its deformations are given by $\bar{\mathfrak{x}}_\iota+\varepsilon\mathfrak{x}_\iota$. Corresponding edges of $(\mathfrak{x})$ and $(\bar{\mathfrak{x}})$ are perpendicular.

An octahedron can only be "flächenstarr" shaky. The $(\mathfrak{x}_\alpha)$ determine two hexadrons $(\mathfrak{y})$ and $(\bar{\mathfrak{y}})$ whose vertices $\mathfrak{y}_\alpha$ and $\bar{\mathfrak{y}}_\alpha$ are associated with the sides of $(\mathfrak{x})$. If $\mathfrak{x}_\iota$ is a vertex of the αth side, $\bar{\mathfrak{x}}_\iota=\bar{\mathfrak{y}}_\alpha+\mathfrak{y}_\alpha\times\mathfrak{x}_\iota$. Any edge of $(\mathfrak{x})$ determines a pair of adjacent sides of $(\mathfrak{x})$ and hence an edge of $(\mathfrak{y})$. Corresponding edges of $\mathfrak{x}$ and $\mathfrak{y}$ are parallel. The 4-edges of $\mathfrak{y}$ and the sides of $(\mathfrak{y})$ are plane. Any hexahedron $(\mathfrak{y})$ with plane 4-edges determines uniquely a shaky octahedron $(\mathfrak{x})$ and its deformations $(\mathfrak{x}_\alpha)$. The $(\mathfrak{y})$'s [the $(\bar{\mathfrak{y}})$'s] are identical with the "flächenstarr" ["eckenstarr"] shaky hexahedrons. Their deformations are given by $\mathfrak{y}_\alpha+\varepsilon\bar{\mathfrak{y}}_\alpha$ $[\bar{\mathfrak{y}}_\alpha+\varepsilon\mathfrak{y}_\alpha]$.

Through $\mathfrak{x}_\iota=\bar{\mathfrak{z}}_\alpha+\mathfrak{z}_\alpha\times\bar{\mathfrak{x}}_\iota$ the deformations of $(\bar{\mathfrak{x}})$ determine two hexahedrons $(\mathfrak{z})$ and $(\bar{\mathfrak{z}})$. The vertices of $(\mathfrak{z})$ [of $(\mathfrak{y})$] are the poles with respect to the unit-sphere of the planes of the 4-edges of $(\mathfrak{y})$[$(\mathfrak{z})$]. The sides and vertices of $(\mathfrak{x})$ are polar to the vertices and sides of still another hexahedron with plane sides which gives rise to another class of six shaky twelve-edges. The total system of twelve shaky twelve-edges is closely analogous to the family of twelve surfaces introduced by Darboux for the study of the infinitesimal deformations of surfaces [Leçons sur la théorie générale des surfaces, vol. 4, Gauthier-Villars, Paris, 1896, pp. 48–86]. *P. Scherk* (Philadelphia, Pa.).

★**Haupt, Otto. Sur quelques problèmes de la théorie des ordres géométriques.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 59–76. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

This colloquium lecture is a discussion of topics selected from the geometry of Juel, all of which have been very greatly generalized along the lines of the author's own work. Geometric order is defined to be a function $f(T)$ over certain subsets T of a topological space R (which was usually a curve in the original Juel theory) which takes values from a well-ordered set (which was the set of non-negative integers in the classical case). This concept leads to a definition of singular points of R.

In nearly all the problems considered the order of T is defined by the use of the number of intersections of T with order characteristics. (In Juel's geometry these were either linear spaces or circles.) A number of results are quoted in which R is located in Euclidean space and each order characteristic is determined by $k+1$ points. One of these is that a curve which satisfies certain monotony conditions with order $\geqq k+2$ contains at least one point of order $\geqq k+2$. Moreover, if such a curve has order $k+2$, then it is the union of at most N arcs of order $k+1$, where N depends only on k. The four-vertex theorem is a consequence of these results. Some unsolved problems are stated.

For finite order curves a complete theory of characteristics is given which is not dependent on differentiability assumptions.

Related problems concerning surfaces are discussed briefly. *D. Derry* (Vancouver, B.C.).

See also: Schläfli, p. 814; Tits, p. 874.

Convex Domains, Extremal Problems, Integral Geometry, Distance Geometry

★**Blaschke, Wilhelm. Kreis und Kugel.** Chelsea Publishing Co., New York, 1949. x+169 pp. Paperbound: $1.50; clothbound: $3.50.

Photo-offset reprint of the edition of 1916 [Veit, Leipzig].

Hadwiger, H.; und Debrunner, H. Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene. Enseignement Math. (2) **1** (1955), 56–89.

This paper consists of the statements and solutions of thirty-seven problems in the elementary combinatorial geometry of the plane together with bibliographical and mathematical comments by the authors. The subject matter, as the authors point out, consists of some of the simplest and most familiar objects of high school geometry, yet many of the questions dealt with lead directly into important and often unsolved problems which are subjects of current research.

The paper is divided into two parts. The first gives the statements of the problems together with the discussion, while the second gives solutions in the form of brief indications of proofs. The first part is further divided into sections in accordance with the more specific nature of the problems. The first section, problems 1–4, is devoted to problems related to the famous conjecture of Sylvester, that if a finite set of points in the plane has the property that every line through two of its points passes through a third then all points lie on a line. Problems 5–9 deal with plane point sets in which the angles or distances determined are rational or integral, e.g. the fact (Erdös) that an infinite plane point set in which all distances are integral must be linear, although there are arbitrarily large finite sets which determine integral distances such that no three points are collinear. Problems 10–13 deal with the notion of convex hull, the basic result being that fact that a point in the convex hull of a set is in the convex hull of some subset of S containing three or fewer points. The section leads into the longest section, problems 14–28, concerned with Helly's theorem, its variations and generalizations. The final section, problems 29–37, deal with the notion of diameter, including the fact that every set of diameter 1 in the plane can be covered by a circle of radius $1/\sqrt{3}$ and by an equilateral triangle of side $\sqrt{3}$.

The proofs of the results are very short, none requiring more than half a page. Although all results are stated only for the plane case, the authors, whenever appropriate, tell whether higher-dimensional analogues are true, false, or unknown. A bibliography containing sixty references at the end of the paper contains the names of many of the foremost geometers of recent times, again indicating the depth of these apparently elementary questions.

The paper constitutes not only a delightful bit of light reading or mathematical recreation for the non-specialist, but also provides a useful reference and source of results and proofs for mathematicians working in this field.

D. Gale (Providence, R.I.).

Lenz, Hanfried. Zur Zerlegung von Punktmengen in solche kleineren Durchmessers. Arch. Math. **6** (1955), 413–416.

Let K be a closed convex subset of n-space of diameter 1 all of whose boundary points are regular (i.e. lie on unique supporting hyperplanes). The author proves: (1) K is the union of $n+1$ subsets of diameter less than 1. (This proves

a special case of the famous conjecture of Borsuk [Fund. Math. **20** (1933), 177–190] and generalizes a result of Hadwiger [Math. Z. **51** (1949), 161–165; MR **10**, 59]. (2) K is the union of fewer than $n+1$ subsets of smaller diameter if and only if K does not have constant breadth. (3) If K has an orthogonal projection onto a k-dimensional subset of diameter less than 1, then K is the union of $n+1-k$ subsets of diameter less than 1.

A boundary point which is not regular is called singular. For general convex sets K of diameter 1 it is shown that if the set of singular points of K is not empty and is the union of k sets of diameter less than 1, then K is the union of $n+k$ sets of diameter less than 1. *D. Gale.*

Lenz, Hanfried. Über die Bedeckung ebener Punktmengen durch solche kleineren Durchmessers. Arch. Math. **7** (1956), 34–40.

Let M be an arbitrary bounded subset of euclidean n-space E_n, and denote by $d_k(M)$ ($d_k'(M)$) the greatest lower bound of numbers x with the property that M is coverable by k subsets of E_n (k spheres of E_n) with diameters at most x. Without restricting the generality, it may be assumed that the covering sets are closed and convex. It is easily seen that M may be covered by k sets (spheres) with diameters at most $d_k(M)$ ($d_k'(M)$). Evidently $d_1(M)$ is the diameter of M and $d_1'(M)$ is the diameter of the circum-sphere of M. The paper is concerned exclusively with plane sets. Some results are: (1) if M is bounded, closed and convex, with diameter 1, then

$$d_3(M)\leqq d_3'(M)\leqq \tfrac{1}{2}\sqrt{3},\ d_4(M)\leqq d_4'(M)\leqq \tfrac{1}{2}\sqrt{2};$$

(2) if M is of constant breadth 1, then $d_3(M)\geqq\sqrt{3}-1$, with equality only when M is a Reuleaux triangle.

L. M. Blumenthal (Columbia, Mo.).

Levi, F. W. Überdeckung eines Eibereiches durch Parallelverschiebung seines offenen Kerns. Arch. Math. **6** (1955), 369–370.

Let Γ be a compact convex subset of the plane. Levi has shown [Arch. Math. **5** (1954), 476–478; MR **16**, 163] that if Γ is a parallelogram, it cannot be covered by fewer than four translates of its interior. It is here shown that in all other cases the number of translates can be reduced to three. *D. Gale* (Providence, R.I.).

Leech, John. The problem of the thirteen spheres. Math. Gaz. **40** (1956), 22–23.

It was conjectured by David Gregory, in an unpublished notebook, that a sphere can touch thirteen non-overlapping spheres equal to it. This conjecture is now known to be false [Schütte and van der Waerden, Math. Ann. **125** (1953), 325–334; MR **14**, 787]. The author gives a new and simpler proof, based on the impossibility of a polyhedron having 13 vertices (1 of degree 4, and 12 of degree 5) and 21 faces (20 triangles and 1 quadrangle).

H S. M. Coxeter (Toronto, Ont.).

★ **Fenchel, W. Sur les variétés localement convexes des espaces projectifs.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 95–104. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

Let V^{n-1} be a compact $(n-1)$-dimensional topological manifold ($n\geqq 3$), and let $\Phi^{n-1}=fV^{n-1}$ be a locally homeomorphic image of V^{n-1} in Π^n, n-dimensional (real) projective space. If Φ^{n-1} is locally strictly convex, then V^{n-1} is homeomorphic to a sphere and Φ^{n-1} is the boundary of a convex domain. Two proofs are given for this theorem: the first, assuming in addition that Φ^{n-1} is simple, reduces the considerations to the covering sphere of Π^n; the second, more involved, proof operates in the original projective space. The analogous theorem for imbedding in Euclidean space has been treated by, among others, J. Van Heijenoort [Comm. Pure Appl. Math. **5** (1952), 223–242; MR **14**, 577]. *L. W. Green.*

Pogorelov, A. V. On the rigidity of general infinite convex surfaces with integral curvature 2π. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 19–20. (Russian)

The paper states the most general rigidity theorem for unbounded convex surfaces: Two intrinsically isometric complete unbounded convex surfaces with integral curvature 2π in E^3 are congruent (i.e. one may be carried into the other by a motion including, of course, reflection).

It is known that surfaces of integral curvature less than 2π are never rigid. A proof is outlined for the special case where the surfaces are strictly convex and possess everywhere tangent planes. Regarding the general case the author says that the proof is considerably more involved. The rigidity of general convex caps, among caps, which follows from the author's rigidity theorem for general convex surfaces [Unique determination of general convex surfaces, Akad. Nauk Ukrain. SSR, Kiev, 1952; MR 16, 162] and the integral which is decisive in Herglotz's rigidity proof (applied to analytic surfaces approximating the given surface) are the principal tools. *H. Busemann.*

★ **Santaló, L. A. Sur la mesure des espaces linéaires qui coupent un corps convexe et problèmes qui s'y rattachent.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 177–190. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

This is an expository paper on the integral geometry of convex bodies in an n-dimensional Euclidean space E_n. Part I is concerned with the measure H_r^n of r-dimensional linear spaces which meet a convex body K^n. If the boundary $\dot{K}^n$ of K^n is a differentiable hypersurface, H_r^n is, up to a numerical factor, equal to the integral over $\dot{K}^n$ of the elementary symmetric function of order $r-1$ in the principal curvatures of $\dot{K}^n$. Two integral formulas for H_r^n are given, one expressing it as an integral of the corresponding measures of the projections of the convex body, over all directions of projection, and the other expressing it as the integral of the corresponding measures of the intersections of the convex body with linear spaces of a given dimension, over all such linear spaces. As a consequence a generalization of Minkowski's inequality is proved. Part II discusses the kinematic density and the kinematic formula and concludes with two inequalities on the number of cubes or spheres with a given side or radius, which are needed to cover a domain homeomorphic with the sphere. *S. Chern* (Chicago, Ill.).

★ **Blaschke, Wilhelm. Vorlesungen über Integralgeometrie. 3te Aufl.** Deutscher Verlag der Wissenschaften, Berlin, 1955. viii+130 pp. DM 13.60.

The changes compared to the earlier edition, are very minor, except in the list of references which has been brought up to date, Two items have been added at the end of § 22 (Aufgaben und Lehrsätze zur Kinematik), one at the end of § 44, and a new section § 45, just over one

page long, deals briefly with cinematic density in non-Euclidean geometry. *H. Busemann.*

Zaharov, D. A. On isotopy of parabolas. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 27–31. (Russian)

Two sets F, F' in metric spaces R, R' are strongly isotopic if a Lipschitz mapping of R on R' exists which takes F into F'. (The mapping $x \to x'$ is Lipschitzian if constants $m>0$, M exist such that the distances xy, $x'y'$ of corresponding pairs of points satisfy $m<xy/x'y'<M$.) A parabola of order n in E^2 is a curve of the form $y=\sum_{i=0}^{n} a_i x^i$, $a_n \neq 0$. Two parabolas of the same order are strongly isotopic. All parabolas of odd order are strongly isotopic to each other (or to a straight line). Two parabolas of different even orders are not strongly isotopic. *H. Busemann* (Los Angeles, Calif.).

See also: Yarutkin, p. 881.

Differential Geometry

Macmillan, R. H. Curves of pursuit. Math. Gaz. **40** (1956), 1–4.

Kahane, Arno. Applications des fonctions de Bessel à l'étude centroaffine des courbes planes. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 93–104. (Romanian. Russian and French summaries)

Bessel functions are used in order to determine the plane curves defined by: $Ks=-n$, where n is a constant, s the "centro-affine arc" and $K(s)$ the "centro-affine curvature". [For these notions see Kahane, Acad. R. P. Române. Stud. Cerc. Mat. **1** (1950), 175–250; MR **12**, 741.] *R. Blum* (Saskatoon, Sask.).

Samorukov, B. N. A congruence of lines associated with a point of a surface. Rostov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 2, 3–8. (Russian)

Take the tangent plane at a point M of the surface S. A fixed point M^* in this plane describes a surface S^* if M describes S. The normals to these surfaces S^*, when M^* varies in the tangent plane, form a congruence. Some properties of the developables and the focal surfaces of this congruence are derived. E.g.: the developable surfaces intersect the tangent plane in two line pencils, the poles of which are in the centers of geodesic curvature of the lines of curvature. *D. J. Struik.*

Šerševskiĭ, E. P. On mappings of surfaces onto a plane for which circles in the plane correspond to geodesic circles of Darboux. Rostov. Gos. Univ. Sb. Stud. Nauč. Rabot. **2** (1953), 64–70. (Russian)

A proof of the theorem that the necessary and sufficient condition that the curves of constant geodesic curvature of a surface can be mapped on the circles of the plane is that the surface be of constant curvature. Some properties of the mapping are studied, especially the mapping of the geodesic lines. *D. J. Struik.*

Cygankov, I. V. On the correspondence of surfaces of orthogonality of line elements and the congruence of Ribaucour. Molotov. Gos. Univ. Uč. Zap. **8**, no. 1 (1953), 15–18. (Russian)

A surface S with quadratic forms

$$Edu^2+2Fdudv+Gdv^2,\ Ddu^2+2D'dudv+D''dv^2$$

is given by the radius vector $\bar{r}(u, v)$. Corresponding by orthogonal line elements is a surface S with radius vector $\bar{r}_1(u, v)$, hence $d\bar{r}\cdot d\bar{r}_1=0$. Then there exists a vector $\bar{l}(u, v)$ such that $\bar{r}_{1u}=\bar{l}\times\bar{r}_u$, $\bar{r}_{1v}=\bar{l}\times\bar{r}_v$, which is the radius vector of a surface Σ. The expressions for $\bar{r}_{1u}$, $\bar{r}_{1v}$ and $\bar{l}$ in terms of $\bar{r}_u$, $\bar{r}_v$ and the normal $\bar{n}$ to S are derived and a number of consequences given, such as the property that S and Σ correspond through parallel normals, that the relation between S and Σ is a reciprocal one, that $\bar{l}$ is normal to S_1, etc. Application to the congruences of Ribaucour is possible through rays passing through points of S_1 parallel to the normals of S or vice versa, since one of the surfaces is then the middle surface of such a congruence. *D. J. Struik* (Cambridge, Mass.).

Samorukov, B. N. A surface of second order tangent to a given ruled surface. Rostov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 2, 49–54. (Russian)

By studying three consecutive generators of a ruled surface with the aid of dual numbers $a+b\omega$, $\omega^2=0$ [here not so much ascribed to Study as to A. P. Kotel'nikov, Calculus of screws, Kazan, 1896] properties are derived for the quadric surface tangent to the ruled surface along a generator. Also quoted in relation to dual numbers is D. N. Zelliger, "Complex line geometry" [Gostehizdat, Leningrad-Moscow, 1934]. *D. J. Struik.*

Buharaev, R. G. On surfaces of Euclidean space with a degenerate absolute. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 39–52. (Russian)

Two conjugate complex points in the plane at infinity of three-dimensional affine space are distinguished, and the differential geometry belonging to the group of affinities leaving these two points fixed is developed. A plane through the two points is called special; the geometry in a special plane is, of course, euclidean. Using the special planes wherever possible is the main tool. For instance, the moving frame in curve theory is chosen such that one vector belonging to $x(t)$ is the tangent vector $\dot{x}(t)$, but the others are two mutually orthogonal vectors in the special plane through $x(t)$, one of which lies in the osculating plane. (The case where $\dot{x}(t)$ lies in a special plane is excluded by hypothesis and similarly in surface theory.) These are analogous to such classical concepts and theorems, like normal curvature, Meusnier's and Euler's theorems, Gauss and mean curvature, the three fundamental forms etc. Some special surfaces, in particular those of constant Gauss curvature, are studied. *H. Busemann* (Los Angeles, Calif.).

Sternberg, Shlomo. On the behavior of invariant curves near a hyperbolic point of a surface transformation. Amer. J. Math. **77** (1955), 526–534.

Transformations of the form S: $x_1=sx+f(x, y)$, $y_1=ty+g(x, y)$ are considered, where (a) $0<s<1$, $s<t$ or, alternatively, (b) $s>1$, $s>t>0$, and where f and g are continuous and are $o(r)$ as $r=(x^2+y^2)^{\frac{1}{2}}\to 0$. Both in case (a) and in case (b), a set E is constructed which is invariant under S. This E will be an arc tangent at 0 to the x-axis, if f and g are uniformly Lipschitz continuous (E otherwise need not be an arc), and if the Lipschitz constant is sufficiently small near 0. In case (b), E will be a differentiable arc, if f and g are differentiable. The methods of the paper are geometrical; they generalize results which were originally due to Poincaré and Hadamard. *A. Douglis* (New York, N.Y.).

Blaschke, Wilhelm. Sui tessuti di superficie. Ann. Mat. Pura Appl. (4) **40** (1955), 35–39.

Let $u_i(x, y, z)=u_i=\text{const}$ $(i=0, 1, 2, 3)$ be four families of surfaces in ordinary space which form a convex web in a given domain R. The condition $\sigma_0+\sigma_1+\sigma_2+\sigma_3=0$ determines the Pfaffian forms $\sigma_i=s_i du_i$ except for a common factor $g(x, y, z)$, that is, except by the transformation $\sigma_i^*=\sigma_i/g$. The Pfaffians $2\tau_1=\sigma_0+\sigma_1=-\sigma_2-\sigma_3$, $2\tau_2=\sigma_0+\sigma_2=-\sigma_1-\sigma_3$, $2\tau_3=\sigma_0+\sigma_3=-\sigma_1-\sigma_2$ are linearly independents and the conditions of integrability of $\sigma_i=s_i du_i$ lead to the equations $d\tau_1=a_1\tau_{23}+b_3\tau_{31}-b_2\tau_{12}$, $d\tau_2=-b_3\tau_{23}+a_2\tau_{31}+b_1\tau_{12}$, $d\tau_3=b_2\tau_{23}-b_1\tau_{12}+a_3\tau_{12}$, where $\tau_{ij}=\tau_i\wedge\tau_j$ and $a_1+a_2+a_3=0$. The Pfaffian $\alpha=a_1\tau_1+a_2\tau_2+a_3\tau_3$ is invariant with respect to the transformations $\sigma_i^*=\sigma_i/g$ and is called the curvature of the web. The Pfaffian $\beta=b_1\tau_1+b_2\tau_2+b_3\tau_3$ transforms according to $\beta^*=\beta-d\log g$, and is called the connection of the web. The conditions $\alpha=0$ and $d\beta=0$ characterize the octahedral and hexagonal webs respectively.

L. A. Santaló (Buenos Aires).

Arf, C. Über den Satz von Dubourdieu. Abh. Math. Sem. Univ. Hamburg **20** (1955), 112–114.

Let $u_0=f(u_1, u_2, u_3)$ be the equation of a 4-web of surfaces. Suppose each of the 3-webs of curves $u_\lambda=\text{const}$ is hexagonal $(\lambda=1, 2, 3)$. Then by Dubourdieu's theorem the web $u_0=\text{const.}$ also is hexagonal. Most of its proofs require $f(u_1, u_2, u_3)$ to have derivatives up to the third order [cf., e.g., Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938, § 19]. A more recent proof by Blaschke assumes the existence of the second-order derivatives [Einführung in die Geometrie der Waben, Birkhäuser, Basel-Stuttgart, 1955, § 10; MR **17**, 780]. The present proof assumes only the existence of the total differential df. [Cf. Blaschke and Bol, loc. cit., p. 50, Problem 12.]

P. Scherk (Philadelphia, Pa.).

Katkov, G. F. On nets of level lines of scalar fields. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **33** (1954), 57–61. (Russian)

Different forms are derived for the differential equation $F_{\alpha\beta}du^\alpha du^\beta=0$ of two scalar fields $\varphi=\varphi(u^1, u^2)$, $\psi=\psi(u^1, u^2)$ on a surface.

D. J. Struik (Cambridge, Mass.).

Efimov, N. V. Investigation of the spherical image of a surface of negative curvature. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 628–630. (Russian)

Consider a simply connected complete surface S of negative curvature in E^3 and class C^3 such that any curve C which tends on S to infinity in the sense of the intrinsic metric, also tends to infinity in the sense of E^3, whenever the directions of the normals of S along C tend to a limit. The spherical image Σ of S will cover the unit sphere multiply and is to be understood as a Riemann surface on the unit sphere. Clearly Σ must have boundary points. The following facts are stated with briefly outlined proofs: If the boundary of Σ has an isolated point (something like a branch point of finite or infinite order), then the least upper bound B of the Gauss curvature of S is zero. If the boundary of Σ has a connected part k and $B<0$, then k is an arc of a great circle. If the boundary of Σ consists of a finite or countable number of connected components, then $B=0$. From these facts statements on S not using Σ can be derived, e.g.: if S lies inside a hyperboloid of one sheet, then $B=0$.

H. Busemann.

Hsiung, Chuan-Chih. On differential geometry of hypersurfaces in the large. Trans. Amer. Math. Soc. **81** (1956), 243–252.

The following theorem is proved. Let V^n (V^{*n}) be an orientable hypersurface of class C^3 imbedded in a Euclidean space E^{n+1} of $n+1\geqq 3$ dimension with a positive Gaussian curvature and a closed boundary V^{n-1} (V^{*n-1}) of dimension $n-1$. Suppose that there is a one-to-one correspondence between the points of the two hypersurfaces V^n, V^{*n} such that at corresponding points they have the same normal vectors and equal sums of the principal radii of curvature, and such that the two boundaries V^{n-1}, V^{*n-1} are congruent. Then the two hypersurfaces are congruent or symmetric. For closed hypersurfaces the theorem is due to T. Kubota [Sci. Rep. Tôhoku Imp. Univ. (1) **14** (1925), 399–402]. The proof is based on the evaluation of the integrals $\int_{V^n} M_\alpha p^* dA$ $(\alpha=1, 2, \cdots, n)$, where M_α is the αth mean curvature of V^n at the point P, p^* the oriented distance from a fixed point 0 in E^{n+1} to the tangent hyperplane of V^{*n} at P^* (corresponding to P) and dA the area element of V^n at P.

L. A. Santaló.

Kovancov, N. I. On a class of surfaces of projective differential geometry. Ukrain. Mat. Ž. **4** (1952), 137–154. (Russian)

This is a careful investigation of the surfaces satisfying the equation

$$(\beta\gamma^2)^{-\frac{1}{4}}\frac{\partial \ln \gamma\beta^2}{\partial v}=(\gamma\beta^2)^{-\frac{1}{4}}\frac{\partial \ln \beta\gamma^2}{\partial u},$$

where $\beta du^3+\gamma dv^3$ is the cubic form of Fubini and the coordinate lines are asymptotic lines. The study is based on the book by S. P. Finikov [Projective differential geometry, Gostehizdat, Moscow-Leningrad, 1937].

D. J. Struik (Cambridge, Mass.).

Rapcsák, András. Eine neue Definition der Normalkoordinaten im Finslerschen Raum. Acta Univ. Debrecen. **1** (1954), 109–116; additamentum ad **1** (1955), 17. (Hungarian. German summary)

In dieser Arbeit wird, in Anlehnung an H. S. Ruse, eine neue Definition der im Finslerschen Raum eingeführten Normalkoordinaten gegeben. O. Varga hat im Finslerschen Raum Normalkoordinaten eingeführt. Diese wurden von ihm mit Hilfe einer Kurvenschar definiert, welche eine kleine Umgebung des Zentrums des Normalkoordinatensystems einfach bedecken. Im Gegensatz zum Punktraum sind diese Kurven im allgemeinen keine geodätische Linien; die geodätische Eigenschaft kommt ihnen nur in speziellen Finslerschen Räumen zu. Wird die partielle Ableitung der Hälfte der quadrierten Bogenlänge mit g_{ik} überschoben, dann ist der sich so ergebende Ausdruck gleich den Normalkoordinaten. Ihr Zentrum ist die Stelle, an der die partielle Ableitung gebildet wurde.

Zusammenfassung des Verfassers.

Varga, Otto. Eine Charakterisierung der Finslerschen Räume mit absolutem Parallelismus der Linienelemente. Acta Univ. Debrecen. **1** (1954), 105–108; additamentum ad **1** (1955), 16. (Hungarian. German summary)

Es wird folgender Satz bewiesen: Notwendig und hinreichend dafür, dass in einem Finslerschen Raum F_n oder einem Teilbereich desselben ein absoluter Parallelismus der Linienelemente existiert, ist neben der unbeschränkten Existenz von Extremalfeldern, die Voraussetzung, dass der jedem Extremalfeldernzugeordnete

oskulierende Riemannsche Raum R_n eine Translationsgruppe gestattet, deren Bahnen die Feldextremale sind.

Zusammenfassung der Verfassers.

See also: Auslander, p. 862; Yarutkin, p. 881; Domšlak, Levinov and Semyanistyĭ, p. 885; Sauer, p. 886; Haupt, p. 887.

Riemannian Geometry, Connections

Kobayashi, Shôshichi. Espaces à connexions affines et Riemanniennes symétriques. Nagoya Math. J. 9 (1955), 25–37.

The first part of this paper discusses the notion of an affine connexion from the classical point of view and from the bundle point of view, including a discussion of symmetric spaces.

The second part of the paper does the following. Let M be a manifold with a given affine connexion on H, the bundle of bases over M. Fix a point $h_0 \in H$ and let H^0 be the sub-bundle of H consisting of all points that can be joined to h_0 by horizontal curves in H. Let the ω_i and ω_{ij} be the usual forms on H, the ω_{ij} depending on (and being equivalent to) the connexion on H. Let $\omega_i{}^0$, $\omega_{ij}{}^0$ be their restrictions to H^0. Define $A(H^0)$ to be the group of all diffeomorphisms of H^0 leaving the $\omega_i{}^0$ and $\omega_{ij}{}^0$ invariant. $A(H^0)$ is a Lie group. There is a natural mapping α of $A(H^0) \to H^0$ defined by: if $\varphi \in A(H^0)$, then $\alpha(\varphi) = \varphi(h_0)$.

The connexion is called locally reductive if its curvature and torsion are invariant under parallel translation. The main new theorem proved states that if M is simply connected and the affine connexion is locally reductive and complete then $A(H^0) = H^0$ under α. The proof goes by defining $L' =$ all vector fields X on H^0 such that all $\omega_i{}^0(X)$ and $\omega_{ij}{}^0(X)$ are constant functions on H^0, $L =$ all vector fields Y on H^0 such that $[Y, L'] = 0$, then proving L is the Lie algebra of $A(H^0)$, and that dimension of $L =$ dimension of H^0. *W. Ambrose* (Cambridge, Mass.).

Nomizu, Katsumi. Reduction theorem for connections and its application to the problem of isotropy and holonomy groups of a Riemannian manifold. Nagoya Math. J. 9 (1955), 57–66.

The main theorem proved here asserts that if M is a complete Riemannian manifold such that at each $m \in M$ the component of the identity H of the holonomy group (as a group of transformations on the tangent space at m) is contained in the linear isotropy group at m then M is symmetric. The linear isotropy group at m is the group of transformation on the tangent space at m induced by the isometries of M which leave m fixed. As one application it is shown that if M is compact, and H is irreducible, and dim (group of isometries of M) = dim H + dim M, then M is symmetric. *W. Ambrose* (Cambridge, Mass.).

Nomizu, Katsumi. Studies on Riemannian homogeneous spaces. Nagoya Math. J. 9 (1955), 43–56.

Let M be a Riemannian manifold. An affine transformation on M is a diffeomorphism of M onto M whose differential commutes with parallel translation along all differentiable curves. Let $A(M)$ denote the group of affine transformations of M, $I(M)$ the group of all isometries of M, and $A^0(M)$, $I^0(M)$ their identity components; $A(M)$ and $I(M)$ are known to be Lie groups.

Let $m \in M$. The holonomy group H at m acts on the tangent space M_m at m. M is called irreducible if H is irreducible on M_m. In general, let $M_m{}^0$ be the points left fixed by H, and $M_m{}^1, \cdots, M_m{}^r$ the irreducible invariant subspaces of the orthogonal complement of $M_m{}^0$. If M is simply connected and complete then de Rham has shown that M can be expressed as a Riemannian product of Riemannian manifolds: $M = M^0 \times \cdots \times M^r$ such that $M_m{}^i$ is the tangent space to M^i at m. This is the de Rham decomposition of M and M^0 is called the Euclidean part of M. Hence every point $n \in M$ has a decomposition of M_n into $M_n{}^0, \cdots, M_n{}^r$.

The "fundamental lemma" of this paper states that $\varphi \in A^0(M)$ implies $d\varphi M_n{}^i = M^i{}_{\varphi(n)}$, i.e. φ preserves the de Rham decomposition. It is then shown that if M is irreducible (not necessarily simply connected or complete) then any compact or connected semi-simple subgroup of $A(M)$, also the commutator subgroup of $A(M)$, is in $I(M)$; if M is simply connected and complete (not necessarily irreducible) and its Euclidean part has dimension ≤ 1 then the corresponding subgroups of $A^0(M)$ are in $I(M)$.

Now let M be a Riemannian homogeneous space; $M = G/H$. It is shown, if M is simply connected, that the factors M^i in any de Rham decomposition have the form: $M^i = G_i/H$ where G_i is a connected closed subgroup of G and contains H. Under certain conditions the holonomy algebra of $M = G/H$ is determined in the following terms. Assume G/H reductive, i.e. $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ with $\mathrm{ad}\, H\mathfrak{m} \subseteq \mathfrak{m}$ ($\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, $\mathfrak{m}$ a subalgebra of $\mathfrak{g}$); hence there is a natural identification of $\mathfrak{m}$ with M_m for any $m \in M$. Through this, covariant differentiation on M at m gives rise, for each $X \in \mathfrak{m}$, to an endomorphism $\varrho(X)$ of $\mathfrak{m}$. Let $\mathfrak{h}_1$ be the ideal in $\mathfrak{h}$ spanned by the projection into $\mathfrak{h}$ of all $[X, Y]$ where X, $Y \in \mathfrak{m}$. Then, under certain assumptions, the holonomy algebra of M is shown to be generated by $\varrho(\mathfrak{m})$ and $\mathrm{ad}\, \mathfrak{h}_1$. *W. Ambrose.*

Hano, Jun-ichi. On affine transformations of a Riemannian manifold. Nagoya Math. J. 9 (1955), 99–109.

We use the notation of the preceding review. If M is a simply connected complete Riemannian manifold and $M = M^0 \times \cdots \times M^r$ is a de Rham decomposition of M it is shown that $A^0(M)$ is naturally isomorphic to the product of the $A^0(M^i)$, and $I^0(M)$ to the product of the $I^0(M^i)$. The proof depends upon the fundamental lemma of the paper reviewed above.

Let M be any manifold and V a vector field on M. Here V gives rise to a local 1-parameter group of local transformations defined by: if $m \in M$ and t is a real number then $\varphi_t(m) =$ point of parameter value t on the integral curve of V which starts at m. Now let M be Riemannian. V is called an infinitesimal affine transformation if the $d\varphi_t$ commute with covariant differentiation; V is called a Killing field if the φ_t are local isometries. If M is complete then an infinitesimal affine transformation gives rise to a global 1-parameter group of affine transformations and a Killing field to a 1-parameter group of isometries. Hence the infinitesimal affine transformations are the Lie algebra of $A^0(M)$ and the Killing fields are the Lie algebra of $I^0(M)$.

It is shown that if M (Riemannian) is irreducible and complete then an infinitesimal affine transformation V is a Killing field if and only if there is a non-trivial orbit, under the corresponding 1-parameter group of affine transformations, on which the length of V is bounded. Also, if M is flat and V is an infinitesimal affine transformation whose length is bounded it is shown that V is a Killing field.

Using these facts and the de Rham decomposition for

the covering space of M it follows that if M is a complete Riemannian manifold and V an infinitesimal affine transformation on M whose length is bounded then V is a Killing field. This implies a theorem of Yano that on a compact Riemannian M every infinitesimal affine transformation is a Killing field and hence $A^0(M)=I^0(M)$.

W. Ambrose (Cambridge, Mass.).

Kobayashi, Shoshichi. A theorem on the affine transformation group of a Riemannian manifold. Nagoya Math. J. **9** (1955), 39–41.

An elegant proof of the theorem: If M is an irreducible and complete Riemannian manifold, then $A(M)=I(M)$ except when M is the real line. [This notation is defined in the second preceding review.] *W. Ambrose.*

Ishihara, Shigeru; and Obata, Morio. On a homogeneous space with invariant affine connection. Proc. Japan Acad. **31** (1955), 421–425.

Some conditions are given under which an invariant affine connexion on a reductive homogeneous space is shown to be a product of two such. Then some other conditions are given under which an affinely connected manifold admitting a transitive group of affine transformations has zero curvature or zero torsion.

W. Ambrose (Cambridge, Mass.).

Ishihara, Shigeru; and Obata, Morio. On the group of conformal transformations of a Riemannian manifold. Proc. Japan Acad. **31** (1955), 426–429.

Let M be a Riemannian manifold of dimension $n\geqq 3$, φ a diffeomorphism of M onto M. φ is called conformal if for each $p\in M$ there is a constant $f(p)$ such that $(G\circ d\varphi)_p=f(p)G_p$, where G is the scalar product of the Riemannian structure; f is called the associated function of φ. φ is called a homothety if f is constant, an isometry if $f=1$. Let $K(M)$ be the group of all conformal transformations of M, $K_p(M)$ the conformal transformations that leave p fixed. If all $d\varphi_p$ for $\varphi\in K_p(M)$ are isometries, then p is called an isometric point, and otherwise p is called a homothetic point. It is proved that Weyl's conformal curvature tensor vanishes at a homothetic point. Among other theorems proved are: 1) If $K(M)$ is transitive and there exists a homothetic point, then M is conformally flat; 2) if M is compact, orientable and connected, then the associated function of any conformal transformation takes the value 1 at some point. *W. Ambrose.*

Širokov, P. A. Symmetric spaces of the first class. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 71–82. (Russian)

This paper by the late P. A. Širokov was prepared for publishing by A. Z. Petrov. Its result is that Riemannian spaces for which $R_{\alpha\beta\gamma\delta,\varepsilon}=0$ and $R_{\alpha\beta\gamma\delta}=h_{\alpha\gamma}h_{\beta\delta}-h_{\alpha\delta}h_{\beta\gamma}$ and which are not of constant curvature and which are irreducible have their fundamental tensor of the form (Riemannian coordinates):

$$g_{\alpha\beta}=a_{\alpha\beta}-\tfrac{1}{3}b_{\alpha\beta}a_{\sigma\tau}x^\sigma x^\tau+\tfrac{1}{3}b_{\alpha\sigma}x^\sigma b_{\beta\tau}x^\tau,$$

the $b_{\alpha\beta}$ being a nilpotent tensor of rank $\frac{1}{4}n$ ($b_\alpha{}^\sigma b_{\sigma\beta}=0$). Then the actual imbedding of such manifolds in a flat $(n+1)$-dimensional space is discussed. *D. J. Struik.*

Takeno, Hyôitirô; and Abe, Shingo. γ-form of the line elements of static spaces. Tensor (N.S.) **5** (1955), 111–122.

A Riemannian space V_n is said to be static if there exists a coordinate system in which the line element takes the form

$$ds^2=g_{ij}dx^idx^j+g_{nn}(dx^n)^2,$$

where $i, j, k=1, 2, \cdots, n-1$, $g_{ij}=g_{ij}(x^k)$, $g_{nn}=g_{nn}(x^k)$. In the present paper the authors give necessary and sufficient conditions for a space to be static and introduce a specialized form for the line element in such spaces. This particular form of line element is used to investigate some of the geometric properties of static spaces and to generalize some of the known results for the case $n=4$.

M. Wyman (Edmonton, Alta.).

Norden, A. P. On a complex representation of tensors of a biplanar space. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 45–53. (Russian)

A biplanar space B_{2n} is endowed with an affinor $G=(G_\alpha{}^\beta)$, $\alpha, \beta, \cdots=1, \cdots, 2n$, for which $G^2+1=0$ (characteristic equation $(G^2+1)^n=0$), which for a well chosen canonical coordinate system takes the form $G_{a+n}{}^a=-G_a{}^{a+n}=1$ $(a, b, \cdots=1, \cdots, n)$; the other components being zero. Transformations from one such system into the other are characterized by $p_\alpha{}^\gamma G_\gamma{}^\beta=p_\gamma{}^\beta G_\alpha{}^\gamma$; such "rotations" take the form

$$(p)=\begin{pmatrix}(p_b{}^a) & (p_{b+n}{}^a)\\ -(p_{b+n}{}^a) & (p_b{}^a)\end{pmatrix}.$$

To a B_{2n} belongs a complex vector space A_n by means of the relation $X^a=x^a+i\tilde{x}^a$, $\tilde{x}^a=G_\beta{}^a x^\beta$ ($\tilde{\tilde{x}}{}^a=-x^a$, similarly $\tilde{\tilde{\xi}}_a=-G_\beta{}^a\xi_\beta$, $\tilde{\tilde{\xi}}_\alpha=-\xi_\alpha$). The group of the biplanar rotations of B_{2n} induces into A_n the general group of nonsingular central affine transformations. The corresponding relation for covectors is $\Xi_a=\xi_a-i\tilde{\xi}_a$. Application is made to the Lorentz space Z_4, $g_{ij}x^ix^j=(x^1)^2+(x^2)^2+(x^3)^2-(x^4)^2$, which through its bivectors can be related to a B_6 of the form considered, and this B_6 again to a complex A_3.

D. J. Struik (Cambridge, Mass.).

Petrov, A. Z. Classification of spaces defining gravitational fields. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 55–69. (Russian)

The author returns to his paper in Dokl. Akad. Nauk SSSR (N.S.) **81** (1951), 149–152 [MR **13**, 581] and enters into more details. In particular, he studies the form of the curvature matrix $R_{\alpha\beta}$, $\alpha, \beta=1, \cdots, 6$, which can be cast into the form

$$\begin{Vmatrix} M & N\\ N & -M\end{Vmatrix},\quad M=\begin{Vmatrix} m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{Vmatrix},\quad N=\begin{Vmatrix} n_{11} & n_{12} & n_{13}\\ n_{21} & n_{22} & n_{23}\\ n_{31} & n_{32} & n_{33}\end{Vmatrix},$$

$m_{pq}=m_{qp}$, $n_{pq}=n_{qp}$, $p, q=1, 2, 3$, $\sum_1^3 m_{pp}=k(R_{ij}=kg_{ij}$, $i, j=1, \cdots, 4)$, $\sum_1^3 n_{qq}=0$. Canonical forms are found for $R_{\alpha\beta}$ in the different types of Einstein spaces.

D. J. Struik (Cambridge, Mass.).

Papy, Georges. Sur la définition intrinsèque des vecteurs tangents à une variété de classe C^r lorsque $1\leq r<\infty$. C. R. Acad. Sci. Paris **242** (1956), 1573–1575.

Continuing the work of an earlier paper [C.R. Acad. Sci. Paris **241** (1955), 19–20; MR **16**, 1152], the author shows that the intrinsic definition of a tangent vector used by Chevalley for analytic manifolds gives an infinite-dimensional tangent space for manifolds of class r, $1\leq r<\infty$. This corrects a statement by the reviewer (loc. cit.). The same result was communicated to the reviewer by Harley Flanders (unpublished). *S. Chern.*

Vrănceanu, G. Sur les espaces A_n non projectivement euclidiens, sans torsion avec group maximum. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 3 (1954), no. 4–5, 87–100. (Romanian. Russian and French summaries)

The author has shown [C.R. Acad. Sci. Paris **229** (1949), 543–545; MR **11**, 134] that a space with affine connection $A_n(x^1, x^2, x^3, \cdots, x^n)$ having non-vanishing projective curvature, admits at most a group of affine motions G_m with $m=n^2-2n+5$ parameters. This maximum number is attained by the Kagan space whose coefficients Γ_{jk}^i are given by:

$$\Gamma_{22}^1=x^3;\quad \Gamma_{jk}^i=0 \text{ (otherwise)}. \tag{1}$$

This result is completed in the present paper. The author proves, that inversely, the only A_n with non-vanishing projective curvature which admits the maximum group of affine motions G_m is the space (1) i.e. every A_n with these properties can be reduced to (1) by a suitable transformation of coordinates.

The following steps lead to the proof of this theorem: 1) The contracted curvature tensors of an A_n of the required type are zero. 2) This A_n is symmetric (in the sense of Cartan) i.e. the covariant derivative of the curvature tensor vanishes. Furthermore, a system of congruences can be found with respect to which the components γ_{bcd}^a of the curvature tensor obtain the canonical form:

$$\gamma_{223}^1=-\gamma_{232}^1=1;\ \gamma_{bcd}^a=0 \text{ (otherwise)}.$$

The proof of the announced theorem is now achieved by considering the transformation formulas of the Γ'_{β} and their reduction to the canonical form (1) by a suitably chosen transformation of coordinates.

Finally, the author considers certain properties "in the large" of the A_n as well as the structure of the maximum group G_m.

This paper is an example for the efficiency of the "calculus of congruences" as a method of investigation of differential-geometrical problems. Due originally to Ricci it has been chiefly developed by the author of the present paper. *R. Blum* (Saskatoon, Sask.).

Postelnicu, Tiberiu. Espaces A_2 à connexion affine linéaire localements euclidiens. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 3 (1954), no. 4–5, 101–131. (Romanian. Russian and French summaries)

This paper is a study of spaces with symmetric affine connection in two dimensions $A_2(x^1, x^2)$ whose coefficients Γ_{jk}^i are linear functions of x^1, x^2 and which are "locally euclidean" i.e. whose curvature tensor vanishes.

In the first part a number of canonical forms for the coefficients Γ_{jk}^i of these spaces is determined. It is pointed out that the reduction to these canonical forms is not invariant, i.e. that they are not necessarily distinct under linear transformations of the coordinates x^1, x^2. A criterion is established in order to determine whether a given connection of the required type can be reduced to one of the above canonical forms.

The second part of this paper is devoted to the problem of the "equivalence in the large" of these spaces (A_2) with the euclidean (E_2). By this is meant the following: Let u^i be a set of cartesian coordinates which are linked to the x^i by the partial differential equations:

$$\frac{\partial^2 u^i}{\partial x^j \partial x^k}+\Gamma_{jk}^i\frac{\partial u^i}{\partial x^l}=0 \quad (i, j, k, l=1, 2).$$

Then A_2 is said to be equivalent in the large with E_2 if, when x^i ranges through A_2, u^i ranges through all of E_2. It turns out that certain canonical forms found in part one of this paper are indeed equivalent in the large with E_2 provided, in general, that some of the intervening constants take on particular values or are otherwise limited in their range. *R. Blum* (Saskatoon, Sask.).

Badea, M. Sur les espaces A_3 à connexion constante. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 4 (1955), no. 6–7, 61–64. (Romanian. Russian and French summaries)

The author establishes a canonical form for the coefficients of a symmetric affine connection in three dimensions if a coordinate system can be found in which these coefficients become constants. *R. Blum.*

Laugwitz, Detlef. Die Vektorübertragungen in der Finslerschen Geometrie und der Wegegeometrie. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 21–28.

The author presents an illuminating comparative survey of the various covariant derivatives in Finsler spaces as defined by Synge [Trans. Amer. Math. Soc. **27** (1925), 61–67], Taylor [ibid. **27** (1925), 246–264] (these two derivatives being identical), Berwald [Math. Z. **25** (1926), 40–73], Cartan [Les espaces de Finsler, Hermann, Paris, 1934], the reviewer [Math. Z. **54** (1951), 115–128; MR **13**, 159], and Barthel [Arch. Math. **4** (1953), 346–354, 355–365; MR **15**, 556]. The latter four derivatives are all distinct. The author considers a system of paths: $\ddot{x}^i+2\Gamma^i(x, \dot{x})=0$, where the Γ^i are homogeneous of degree 2 in the $\dot{x}^i$. The derivatives of the Γ^i with respect to $\dot{x}^i$ give rise to three distinct types of parallelism: if the paths are identified with the geodesics of a Finsler metric, these reduce to the distinct parallelisms of Berwald, Barthel and the reviewer (the latter two differing by the choice of directional argument to be substituted in the first derivatives of the Γ^i). It is pointed out that it is impossible to derive the parallelisms of Taylor, Synge and Cartan in this manner (these being based on the notion of element of support, while Barthel and the reviewer treat Finsler spaces as locally Minkowskian spaces). By using the method of osculating Riemannian spaces two distinct coefficients of the affine connection are deduced from the parallelisms of Barthel and the reviewer. [Remark: The latter coefficients result directly from a study of curvilinear coordinates in a Minkowskian space in a manner analogous to the derivation of the Christoffel symbols of Riemannian geometry from a consideration of curvilinear coordinates in a euclidean space.] *H. Rund.*

Complex Manifolds

Stoll, Wilhelm. Die beiden Hauptsätze der Wertverteilungstheorie bei Funktionen mehrerer komplexer Veränderlichen. I. Acta Math. **90** (1953), 1–115.

L'auteur a pour but de généraliser, aux fonctions de plusieurs variables complexes, la théorie de la répartition des valeurs d'une fonction méromorphe [voir R. Nevanlinna, Eindeutige analytische Funktionen, 2te Aufl., Springer, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 208]. En fait, il étudie des fonctions méromorphes vectorielles, ce qui l'amène à généraliser la théorie des courbes méromorphes et les deux théorèmes fondamentaux de cette théorie [voir H. Weyl, Meromorphic functions and analytic curves, Princeton, 1943; MR **5**, 94].

Soient $\mathfrak{M}$ une variété analytique complexe de dimension réelle $2n$ et un recouvrement ouvert $\{U_j\}$ de $\mathfrak{M}$. Une surface méromorphe W sur $\mathfrak{M}$ est un ensemble de fonctions vectorielles méromorphes $\mathfrak{w}_j(P)$ à k dimensions ($P \in U_j$) telles que, sur $U_j \cap U_k \neq \emptyset$, on ait $\mathfrak{w}_j(P)=\lambda_{jk}(P)\mathfrak{w}_k(P)$ où $\lambda_{jk} \not\equiv 0$ est méromorphe sur $U_j \cap U_k$; $\mathfrak{w}_j(P)$ s'appelle une représentation de la surface méromorphe W sur U_j. La représentation est analytique en $P_0 \in U_j$ quand $\mathfrak{w}_j(P)$ est analytique; elle est, en outre, réduite en P_0 quand les composantes de $\mathfrak{w}_j$ n'ont pas de facteur commun nul en P_0. Les zéros d'une représentation réduite sont les points d'indétermination; en dehors de ces points, W définit une application de $\mathfrak{M}$ dans un espace projectif complexe $\mathfrak{P}$, de dimension réelle $2k-2$, où les coordonnées homogènes sont les composantes du vecteur $\mathfrak{w}_j$.

Dans la première partie de l'article, l'auteur démontre le premier théorème fondamental. Soit ${}^*\mathfrak{P}$ l'espace projectif dual de $\mathfrak{P}$; pour $\mathfrak{w} \in \mathfrak{P}$, $\alpha \in {}^*\mathfrak{P}$, on a le produit scalaire $(\mathfrak{w}, \alpha)=\sum_{\mu=1}^{k} w_\mu \alpha_\mu$. Soit $\nu(P_0, \alpha)$ la multiplicité du zéro P_0 de la fonction $(\mathfrak{w}(P), \alpha)$ où $\mathfrak{w}(P)$ est une représentation irréductible de W en P_0; l'ensemble analytique $\mathfrak{N}(\alpha)=\{P|\nu(P, \alpha)>0\}$ a pour image, par $\mathfrak{w}(P)$, l'intersection de l'image de $\mathfrak{M}$ et de l'hyperplan $(\mathfrak{w}, \alpha)=0$. Soit $\partial\chi$ une $(2n-2)$-forme différentielle fermée sur $\mathfrak{M}$ qui définit, sur toute $(2n-2)$-sous-variété de $\mathfrak{M}$, une densité positive; si $\mathfrak{M}$ est une variété kählérienne dont la forme différentielle associée à la métrique est ∂s_2, on peut choisir $\partial\chi=(\partial s_2)^{n-1}$. On se donne, dans $\mathfrak{M}$, un ouvert relativement compact g (noyau) dont le bord γ est une variété différentiable. On appelle ensemble admissible tout ouvert G relativement compact contenant $\bar{g}$ dont la frontière est une variété différentiable Γ (sauf sur un ensemble de mesure de Carathéodory nulle). Si, pour toute fonction φ, on pose $\partial^\wedge\varphi=i\sum_{\mu=1}^{n}(\varphi_{z_\mu}dz_\mu-\varphi_{\bar{z}_\mu}d\bar{z}_\mu)$, soit $\psi(P, G)$ une fontion satisfaisant à $d\partial^\vee\psi\partial\chi=0$ dans $G-\bar{g}$, nulle en dehors de G et égale à $R(G)$ sur $\bar{g}$, où $R(G)$ est la tension définie de façon unique par la condition $(2\pi)^{-1}\int_\gamma \partial^\wedge\psi\partial\chi=1$. On considère les fonctions suivantes $N(G, \alpha)=\int \nu(P, \alpha)\psi(P, G)\partial\chi$, où l'intégrale est calculée sur $\mathfrak{N}(\alpha)$ et $m(C, \alpha)=-(2\pi)^{-1}\int_C \log\|\mathfrak{w}(P), \alpha\|\partial^\wedge\psi\partial\chi$ où $\|\mathfrak{w}(P), \alpha\| = |(\mathfrak{w}(P), \alpha)|/|\mathfrak{w}(P)\|\alpha|$. Théorème 1. La caractéristique $T(G)=N(G, \alpha)+m(\Gamma, \alpha)-m(\gamma, \alpha)$ est indépendante de α. La démonstration repose sur une généralisation de la formule classique de Jensen. L'auteur termine par différentes représentations et propriétés de $T(G)$, montrant, en particulier, que la caractéristique constitue une mesure de la "croissance" de la surface méromorphe.

P. Dolbeault (Paris).

Stoll, Wilhelm. Die beiden Hauptsätze der Wertverteilungstheorie bei Funktionen mehrerer komplexer Veränderlichen. II. Acta Math. **92** (1954), 55–169.

L'auteur démontre maintenant le second théorème fondamental. Soit $Y \not\equiv 0$ une densité méromorphe; soient $\nu(P)$ la multiplicité de P en tant que zéro de Y moins sa multiplicité en tant que pôle, et $\mathfrak{N}$ la réunion des ensembles des zéros et des pôles de Y. Posons $\Delta=\sum_{\mu,\nu=1}^{n} a_{\mu\nu}\psi_{z_\mu}\psi_{\bar{z}_\nu}$ où les quantités $2^{-n-1}i^{n-1}a_{\mu\nu}$ sont les coefficients de la forme différentielle $\partial\chi$ (définie dans la partie I); alors

$$\eta(G)=\int_{\mathfrak{M}} \nu\psi\partial\chi+\left(\frac{1}{4\pi}\int_\Gamma \log\,\Delta|Y|^{-2}\partial^\wedge\psi\partial\chi-\frac{1}{4\pi}\int_\gamma \log\,\Delta|Y|^{-2}\partial^\wedge\psi\partial\chi\right.$$

est indépendant du choix de la densité Y.

Théorème 2. Hypothèses: 1) Pour tout $\varepsilon>0$ et „la plupart" des ensembles admissibles G, on a $\eta(G)\leqq\varepsilon T(G)$; 2) si J désigne la plus petite borne supérieure (finie ou non) de $R(G)$, alors $\log(1/(J-R(G)))\leqq\varepsilon T(G)$; 3) il existe une $(n-1)$-forme différentielle extérieure analytique

$$\partial B=\sum_{\nu=1}^{n}(-1)^{\nu-1}b_\nu dz_1\wedge\cdots\wedge dz_{\nu-1}\wedge dz_{\nu+1}\wedge\cdots\wedge dz_n$$

telle que

$$\tfrac{1}{4}\sum_{\mu,\nu=1}^{n} a_{\mu\nu}x_\mu\bar{x}_\nu \geqq \sum_{\mu,\nu=1}^{n} b_\mu\bar{b}_\nu x_\mu\bar{x}_\nu;$$

4) certaines expressions construites avec ∂B et la surface méromorphe ne sont pas identiquement nulles; 5) il n'y a pas k vecteurs linéairement indépendants parmi les q vecteurs donnés $\alpha_1, \cdots, \alpha_q$. Conclusion: pour tout $\varepsilon>0$ et „la plupart" des ouverts admissibles G, on a

$$\sum_{\nu=1}^{q} m(\Gamma, \alpha_\nu)\leqq(k+\varepsilon)T(G).$$

Soit $\delta(\alpha)$ la limite inférieure des rapports $m(\Gamma, \alpha)/T(G)$ quand $G\to\mathfrak{M}$; on a: $0\leqq\delta(\alpha)\leqq 1$ et $\sum_{\mu=1}^{q}\delta(\alpha_\mu)\leqq k$; de plus, si la surface méromorphe ne coupe pas l'hyperplan $(\mathfrak{w}, \alpha)=0$, on a $\delta(\alpha)=1$, d'où: parmi $k+1$ hyperplans en position générale, il y en a au moins un qui rencontre la surface méromorphe; ce théoréme généralise le théorème de Picard sur la répartition des valeurs d'une fonction méromorphe.

L'établissement du théorème nécessite le choix d'une forme différentielle ∂B; cette forme intervient dans la construction des „surfaces méromorphes associées" (généralisant les surfaces associées de la théorie de Weyl) qu'il est indispensable d'introduire dans la démonstration. Le théorème n'est pas complètement indépendant du choix de ∂B; dans le cas d'une seule variable, il était connu avec $\partial B=1$. On peut l'améliorer par l'introduction d'un terme $V(G)\geqq 0$ dépendant de G dans le premier membre de l'inégalité et il en existe une généralisation aux surfaces associées.

A la fin de l'article, l'auteur indique la forme que prennent les deux théorèmes lorsque $\mathfrak{M}$ est l'espace euclidien et il retrouve ainsi des résultats de H. Kneser [Jber. Deutsch. Math. Verein. **48** (1938), Abt. 1, 1–28] et de lui-même [Math. Z. **57** (1953), 211–237; MR **14**, 970].

P. Dolbeault (Paris).

Atiyah, M. F. Complex fibre bundles and ruled surfaces. Proc. London Math. Soc. (3) **5** (1955), 407–434.

Cet article traite de la classification des espaces fibrés analytiques complexes dont la base X est une variété algébrique (et plus particulièrement une courbe algébrique), le groupe structural étant l'un des groupes A (affine à une variable) ou P (projectif à une variable).

Soit C le groupe additif des nombres complexes, C^* le groupe multiplicatif des nombres complexes $\neq 0$; le fait que A est produit croisé de C et C^* (C^* opérant sur C par les homothéties) permet de définir: 1^0 une application π de l'ensemble des classes de A-espaces fibrés dans le groupe des classes de C^*-espaces fibrés; 2^0 une application β du groupe de C^*-espaces fibrés dans l'ensemble des classes de A-espaces fibrés; et $\pi\circ\beta$=identité. Etant donné un C^*-espace fibré ξ (de base X), l'ensemble des A-espaces η (de base X) tels que $\pi(\eta)=\xi$ se compose de $\beta(\xi)$ et d'un ensemble de A-espaces (dit „propres") qui est en correspondance biunivoque avec les points d'un espace projectif complexe $Y(\xi)$; la dimension de $Y(\xi)$ est inférieure d'une unité à celle de l'espace vectoriel de cohomologie

$H^1(X, \Omega(\xi))$, $\Omega(\xi)$ désignant le faisceau des germes de sections holomorphes de l'espace ξ. Ceci résout, en principe, le problème de la classification des A-espaces fibrés; si X est une variété algébrique projective, on sait que les ξ correspondent biunivoquement aux classes de diviseurs de X.

Toute classe de A-espaces fibrés définit une classe de P-espaces fibrés. On suppose désormais que X est une courbe algébrique; alors tout P-espace provient d'au moins un A-espace, et le problème de la classification des P-espaces (qui équivaut à la classification des surfaces réglées) sera résolu si l'on sait reconnaître quand deux A-espaces sont P-équivalents. Ceci est ramené à un problème concernant les courbes dans un espace projectif, et on donne une solution explicite dans le cas où la courbe X est de genre $p=0$, 1 ou 2. Si $p=0$, tout P-espace provient d'un C^*-espace; les classes de P-espaces sont en correspondance biunivoque avec les entiers $\geqq 0$ (à chaque classe de diviseurs associer la valeur absolue de son degré); si $p=1$, un C^*-espace et un A-espace propre ne sont jamais P-équivalents, et les A-espaces propres définissent seulement deux classes de P-espaces, qui sont explicitées. Pour $p=2$ les résultats sont plus compliqués.

L'auteur développe aussi des considérations sur les espaces (de base X, de fibre la droite projective) qui peuvent être définis par une application de X dans la grassmannienne G_N des droites de l'espace projectif S_N; et il donne des contre-exemples à des résultats annoncés par Hawley [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 411–415; MR **14**, 73]. *H. Cartan* (Paris).

Goldberg, S. I. Note on projectively Euclidean Hermitian manifolds. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 128–130.

The author considers complex analytic manifolds with Hermitian metric $ds^2=g_{\alpha\beta^*}dz^\alpha d\bar{z}^\beta$. This metric is said to have "constant" holomorphic curvature if $R_{\alpha\beta^*\gamma\delta^*}=\frac{1}{2}K(g_{\alpha\beta^*}g_{\gamma\delta^*}+g_{\alpha\delta^*}g_{\gamma\beta^*})$, and further, in this case, to be flat if $K\equiv 0$. The principal theorem states that a manifold has "constant" holomorphic curvature if and only if it can be transformed to a flat manifold by a special type of projective transformation (geodesics and torsion of the connection preserved). Such manifolds are further asserted to be projectively flat and conformal to a Kähler space.

W. M. Boothby (Evanston, Ill.).

Ishihara, Shigeru. Groups of isometries of pseudo-Hermitian spaces. II. Proc. Japan Acad. **31** (1955), 418–420.

This paper completes results in part I [same Proc. **30** (1954), 940–945; MR **17**, 85]. In particular, the case $n=3$ is now included in the following theorem. If G/H is a homogeneous pseudo-Hermitian space of dimension $2n$ and $\dim G=n^2+2n-1$, then G/H is flat and homeomorphic to E^{2n} or, in case $n=3$, has positive constant curvature, in which case it is homeomorphic to S^6 and G is an exceptional compact simple group. Some other improvements of the previous paper are also included.

W. Ambrose (Cambridge, Mass.).

Algebraic Geometry

Neville, E. H. The power of a point for a curve. Math. Gaz. **40** (1956), 11–14.

Canetta, Pietro. Sulla esistenza di una curva piana algebrica di ordine 8 con 14 cuspidi e 2 nodi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 478–480.

Partant de la tresse relative à la courbe du 8ème ordre dotée de 9 cuspides et de douze points doubles [C. Tibiletti, mêmes Rend. (8) **12** (1952), 537–543; MR **14**, 402], l'A. démontre l'existence d'une octique dotée de 14 cuspides et 2 noeuds. Il en construit la tresse par changement de l'ordre de certains lacets et fusion de certains traits.

B. d'Orgeval (Dijon).

Vaccaro, Giuseppe. Sui sistemi lineari di cubiche per 5 punti di cui 3 allineati. Rend. Sem. Mat. Univ. Padova **24** (1955), 287–299.

En reprenant l'étude d'une surface F du 4ème ordre à conique double de l'espace ordinaire, qui se trouve dans le volume de F. Conforto sur „Le superficie razionali" [Zanichelli, Bologna, 1939 1945], l'A. s'occupe de la construction du système linéaire de courbes planes du 3ème ordre, qui représente F sur un plan, dans le cas où la conique double de F est cuspidale. On a alors un système de cubiques planes avec 5 points de base, dont trois sont alignés et les deux autres coincident. L'A. trouve directement (l'équation de) ce système, tandis que F. Conforto le déduisait d'un système de courbes du 4ème ordre par une transformation quadratique. Après cela, il considère des cas particuliers où F possède une droite cuspidale ou tacnodale.

E. G. Togliatti (Gênes).

Brauner, H. Über die Projektion mittels der Sehnen einer Raumkurve 3. Ordnung. Monatsh. Math. **59** (1955), 258–273.

The present paper deals thoroughly with a birational representation of the chords of a twisted cubic curve by the points of a plane [such a representation had already been partially studied in a paper by G. Aprile, Giorn. Mat. Battaglini (3) **10(57)** (1919), 129–150, cap. VI, § 2, which, however, has been here overlooked]. This leads to a concrete deduction of the well known representation of the linear complexes of lines of a three-dimensional space by the conics of a plane [to the bibliography on this subject given by the author should be added C. Segre, Atti Acad. Sci. Torino **20** (1885), 487–504], as well as to several other applications.

B. Segre (Rome).

Marchionna Tibiletti, Cesarina. La irregolarità di un piano multiplo dedotta dalla treccia diramante. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 480–486.

Une tresse algébrique est dite diramante, si on associe à chacun de ses fils, la substitution relative aux déterminations d'une $z(x, y)$ définissant un plan multiple F. Le calcul de l'irrégularité de ce plan multiple se raméne à celui de son indice de connexion linéaire. Si p est le genre de la section f_0 par $x=0$, l'irrégularité sera $q=p-r/2$, r étant la caractéristique de la matrice $\|[\gamma_i, \delta_k]\|$ où $\gamma_1, \cdots, \gamma_{2p}$ sont $2p$ cycles indépendants de f_0, δ_k les cycles d'une riemannienne S_0 de f_0, $[\gamma_i\delta_k]$ l'indice de Kronecker; les δ sont définis par rapport au point critique a_i où coïncident y_1 et y_2, comme homologues au couple de lacets c_1' et c_2' amenés par continuité sur S_0. Les γ sont aussi exprimés en fonction de ces lacets, d'où le calcul. Le calcul est effectué sur la tresse d'une octique à 9 cuspides diramantes d'un plan triple, alors $q=1$.

B. d'Orgeval (Dijon).

Chisini, Oscar. Il teorema d'esistenza delle trecce algebriche. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 8–13.

[Pour les parts I–II voir mêmes Rend. (8) **17** (1954), 143–147, 307–311; MR **17**, 86.] Si on considère une courbe C_n^r ayant un point r-ple en Y_∞, on peut lui associer une tresse amplifiée, dont la forme canonique s'obtient à partir d'une C_n et de r droites passant en Y_∞, a_i, en conservant les points doubles intersections des droites a_k et des rameaux de C_n autres qu'un rameau fixé, par exemple 1; Y_∞ restant r-ple et faisant disparaître les noeuds D_{kj}; sur la tresse ainsi construite se succéderont les lacets qui vont au pôle z_i (abscisse du point Z_i de l'axe des x où passe l'asymptote de C_n^r proche de a_i), puis ceux qui vont aux abscisses des D_{ij}. Si dans une courbe variable, deux points de diramation se confondent, les lacets venant à coïncider, deux traits de la tresse se confondent: une telle opération faite sur une tresse est dite fusion. L'introduction de la tresse amplifiée et de l'opération de fusion permet de démontrer qu'il existe une courbe d'ordre n effectivement construite, à partir d'une, dont on connait la tresse, par des opérations correspondant à l'introduction de nouveaux noeuds et cuspides pourvu que le nombre de ceux-ci vérifie $d+2k \leqq \frac{1}{2}n(n+3)-3$. Ceci peut s'étendre à des singularités plus compliquées.

B. d'Orgeval (Dijon).

Godeaux, Lucien. Remarques sur les involutions cycliques appartenant à une variété algébrique à trois dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 108–113.

Godeaux, Lucien. Sur la surface des couples de points de la quintique de Snyder. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 8–10.

Spampinato, Nicolò. Rappresentazioni complesse di superficie biduali prolungamento di superficie complesse. Ricerche Mat. **4** (1955), 30–47.

La considerazione di una superficie algebrica biduale ottenuta come prolungamento nel campo biduale di una superficie algebrica complessa f equivale ad associare ad ogni forma $f(x_1, x_2, x_3, x_4)$ la forma polare $\Delta f = \sum y_i \partial f/\partial x_i$. L'Autore studia la V_5 rappresentata nell' $S_7(x_i, y_i)$ dalle equazioni: $f=\Delta f=0$, e la V_6 di S_7 descritta da detta V_5 al variare di f in un fascio lineare, insieme a varie questioni che a ciò si collegano. Tra l'altro studia le trasformazioni birazionali di S_7 che dànno le rappresentazioni complesse delle trasformazioni birazionali biduali ottenute prolungando nel campo biduale le trasformazioni birazionali complesse dell'S_3 complesso. *D. Gallarati.*

Engel, Wolfgang. Primdivisoren höherer Art und ihr Verhalten bei Cremona-Transformationen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **4** (1954/55), 1203–1213.

Generalities on power series expansions for functions of several variables. If V^n is a variety over k and v an $(n-1)$-dimensional valuation of a finite algebraic extension of $k(V)$, we say that v is of ith type with respect to V if its center on V has codimension i. Valuations which change type under a birational transformation must have their centers on one of the fundamental loci involved.

M. Rosenlicht (Evanston, Ill.).

Severi, Francesco. Complementi alla teoria delle equivalenze sulle varietà algebriche: le equivalenze algebriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 357–361.

Deux cycles (=variétés virtuelles) A, B sur une variété V sont dits algébriquement équivalents si $A-B$ est la différence de deux cycles d'un même système algébrique irréductible; énoncé et démonstration de deux autres caractérisations de l'équivalence algébrique. Deux cycles positifs qui font partie d'un même système algébrique connexe de cycles positifs sont algébriquement équivalents; un exemple montre que la réciproque est inexacte.

P. Samuel (Clermont-Ferrand).

Severi, Francesco. Complementi alla teoria delle equivalenze sulle varietà algebriche: le equivalenze razionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 443–451.

Définition et caractérisations de l'équivalence rationnelle, analogues à celles données ci dessus pour l'équivalence algébrique, les systèmes algébriques étant remplacés par les systèmes rationnels (c.à.d. paramétrés par des variétés rationnelles). On appelle système élémentaire projectif de dimension k sur une variété M^r d'un espace projectif P^n tout système (rationnel) découpé sur M^r par les intersections de $r-k$ hypersurfaces de degrés donnés de P^n; un système élémentaire simple de dimension k sur M_r est (à l'addition ou à la soustraction près de composantes fixes) l'ensemble des intersections de $r-k$ diviseurs sur M^r qui parcourent des systèmes linéaires donnés. A une composante fixe près, tout système élémentaire simple est contenu dans un système élémentaire projectif. Tout système unirationnel est combinaison linéaire de systèmes élémentaires; en particulier c'est un système rationnel, et l'équivalence unirationnelle coïncide avec l'équivalence rationnelle. *P. Samuel.*

Severi, Francesco. Sugli antigeneri d'una varietà algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 131–140.

Soit V une variété non singulière de dimension n. Pour tout système linéaire $|C|$ de diviseurs positifs sur V, notons $|C'|$ son système adjoint; le système linéaire $|C-C'|$ est indépendant de $|C|$; on l'appelle le système anticanonique de V et sa dimension s'appelle l'antigenre de V. Il peut arriver, lorsque le système canonique $|K| = |C'-C|$ est vide (au sens des diviseurs positifs), que $|C-C'|$ ne le soit pas. Les surfaces rationnelles normales à sections planes rationnelles ont leur antigenre a égal à 9, sauf le plan et la surface de Veronese pour lesquelles $a=10$; détermination des antigenres d'autres surfaces simples. L'antigenre est un invariant birationnel relatif (c.à.d. birégulier), mais non absolu; l'antigenre absolu d'une classe de surfaces birationnellement équivalentes est le maximum (qui est fini) des antigenres des surfaces de la classe. Cet antigenre absolu est égal à 10 pour les surfaces rationnelles, et ceci caractérise les surfaces rationnelles parmi les surfaces régulières. *P. Samuel.*

Kähler, Erich. Tensori razionali di I^a specie sopra una varietà algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 151–154.

Soient V une variété algébrique non singulière, K son corps de fonctions, $(x_1, \cdots, x_n)$ une base de transcendance séparante de K. Un tenseur rationnel sur V (par exemple 2 fois contravariant et 1 fois covariant) est défini par la donnée de n^3 éléments A_{ij}^k de K, appelés ses composantes

par rapport à $(x_1, \cdots, x_n)$. Ses composantes par rapport à une autre base de transcendance séparante $(y_1, \cdots, y_n)$ de K sont, par définition,

$$B_{ij}^{k} = \sum_{i',j',k'} A_{i'j'}^{k'} \frac{\partial x_{i'}}{\partial y_i} \cdot \frac{\partial x_{j'}}{\partial y_j} \cdot \frac{\partial y_k}{\partial x_{k'}}.$$

On dit qu'un tenseur rationnel sur V est de première espèce si, pour tout point P de V et tout système $(x_1, \cdots, x_n)$ de coordonnées locales (=paramètres uniformisants) en P $(x_i \in K)$, les composantes de ce tenseur par rapport à $(x_1, \cdots, x_n)$ sont dans l'anneau local de P. Une forme anticanonique est un tenseur covariant antisymétrique de degré n; il a essentiellement une seule composante, laquelle se transforme au moyen de $B = A\partial(y_1, \cdots, y_n)/\partial(x_1, \cdots, x_n)$. La dimension de l'espace vectoriel des formes anticanoniques est égal à l'antigenre introduit ci dessus par Severi. *P. Samuel.*

★**Roth, L. Algebraic threefolds, with special regard to problems of rationality.** Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 6. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+142 pp. DM 19.80.

This book gives an account of the birational properties of algebraic threefolds. The emphasis is very much on the algebro-geometric treatment, though for a number of results reference is made to transcendental and topological methods and to theorems of Lefschetz. The very recent work of Kodaira and Hirzebruch is only briefly mentioned. The first three chapters deal with general theory.

Chapter I centers round the invariants defined by the canonical system, the arithmetic genus and irregularities, and the expression of these invariants in terms of projective characters. Chapter II develops the theory of systems of equivalence on a V_3, and the theory of the invariant and covariant systems due to Segre, and concludes with an account of the extension of those notions to varieties of higher dimension. Chapter III deals with the Riemann-Roch theorem for threefolds, and with the theory of the base. The main part of the work is the discussion of various aspects of the classification problem for algebraic threefolds.

Chapter IV deals with questions of rationality and unirationality, more especially with results concerning threefolds containing systems of rational elliptic or hyperelliptic curves, or systems of rational surfaces. Chapter V is concerned largely with threefolds on which the operation of successive adjunction terminates, and gives a detailed account of the Fano threefolds (completely regular threefolds whose curve sections are canonical curves). Chapter VI is concerned with threefolds with infinite groups of birational self-transformations, a topic to which the author has contributed much of what is known. An appendix reviews briefly the necessary background material in the theory of curves and surfaces, and there is an extensive bibliography. *J. A. Todd.*

Barsotti, Iacopo. Un teorema di struttura per le varietà gruppali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 43–50.

Cette note apporte divers compléments à un mémoire récent de l'auteur [Ann. Mat. Pura Appl. (4) **38** (1955), 77–119; MR **17**, 193; même terminologie et notations que dans cette analyse]. Etant donné un groupe algébrique (rappelons que c'est, entre autres, un ouvert d'une variété projective), l'ensemble R_G des fonctions rationnelles partout régulières sur G est un sous-anneau (intégralement clos) du corps $k(G)$ des fonctions de G; le corps des fractions de R_G est le corps $k(A)$ des fonctions d'un groupe de Vessiot A, et l'immersion de $k(A)$ dans $k(G)$ définit un homomorphisme séparable de G sur A dont le noyau est irréductible; le groupe A est „universel" pour les homomorphismes de G dans les groupes de Vessiot. Toute extension G d'un groupe de Vessiot G/V par un sous-groupe rationnel irréductible V du centre de G est un groupe de Vessiot. Tout groupe algébrique contient un sous-groupe de Vessiot invariant H tel que G/H soit une variété abélienne; le sous groupe H est le plus grand groupe de Vessiot contenu dans G; le groupe G se construit, essentiellement, à partir de H, de G/H et d'un système de facteurs. *P. Samuel* (Clermont-Ferrand).

See also: Terraicni, p. 820; Brown, p. 886; Atiyah, p. 894.

NUMERICAL ANALYSIS

★**Nielsen, Kaj L. Methods in numerical analysis.** The Macmillan Company, New York, 1956. xiii+382 pp. $6.90.

This volume gives an account of numerical analysis from an elementary and practical standpoint, emphasis being placed on methods suitable for desk calculating machines. The main topics are finite-difference tables, interpolation, numerical differentiation and integration, differential equations, ordinary and simultaneous equations, and curve fitting. The book is well supplied with examples and exercises of a straightforward nature.

After a preliminary chapter giving the basic concepts required, the author considers the formation of difference tables and their use in the tabulation of polynomials and the location of errors. The next two chapters are devoted to methods involving differences. Firstly the classical interpolation formulae of Newton-Gregory, Stirling, Everett and Bessel are derived with the error terms considered. Sections on divided differences, Aitkens process and bivariate interpolation follow. Formulae for numerical differentiation and integration derived from the classical interpolation formulae are established, and other methods, including Simpson's and Weddle's rules and the method of Gauss are considered. A chapter is then devoted to the use of Lagrangian formulae.

The solution of ordinary equations by iteration and recurrence methods is next considered, including the determination of complex roots. Graeffe's root-squaring process is briefly described. For simultaneous equations, the method recommended is that of Crout, and its application to matrix inversion and the evaluation of determinants is discussed. A section is devoted to the solution of systems of non-linear equations, but the examples are all restricted to two equations.

A chapter on the solution of ordinary and partial differential equations follows, in which only the outlines of the topic are discussed. The methods of Euler, Milne and Runge-Kutta suffice for ordinary differential equations. The section on partial differential equations is intended to be an introduction, but no indication of difficulties or growth of error is considered. The methods outlined are iteration (or Liebmann's process) and relaxation for elliptic equations and step-by-step methods for parabolic and hyperbolic equations.

The application of the method of least squares to polynomial fitting is next considered, and various methods to assist in the calculations including simplifications when evenly spaced data is available, the use of orthogonal polynomials, and the Nielsen-Goldstein method for equally spaced data which generates in the normal equations a matrix of coefficients whose inverse has been found for polynomials of different degree and different numbers of observations [Nielsen and Goldstein, J. Math. Phys. **26** (1947), 120–132; MR **9**, 104]. Further topics considered are the smoothing of data using formulae based on polynomials of various degree, the use of weighted residuals, methods for harmonic analysis, and the fitting of complicated functions for which least squares may yield non-linear equations. Finally the calculation of the autocorrelation function is described.

The book also includes certain tables, the principal ones being tables of interpolation, coefficients at interval 0.01 to 7 decimals, the coefficients of the formulae for differentiation and integration, and tables to assist in curve fitting, including tables of the orthogonal polynomials and those required by the Nielsen-Goldstein method.

D. C. Gilles (Manchester).

Perry, N. C.; and Morelock, J. C. On the propagation of error by multiplication. Amer. Math. Monthly **63** (1956), 177–179.

Plainevaux, J. E. Zur graphischen Konstruktion von rationalen Polynomen. Z. Angew. Math. Mech. **36** (1956), 78.

Salzer, Herbert E. Formulas for inverse osculatory interpolation. J. Res. Nat. Bur. Standards **56** (1956), 51–54.

When both a function and its first derivative are tabulated, or when the latter is easily obtained, the use of Hermite's formula will generally give a more accurate result, for direct interpolation, than that of Lagrange. Inversion of Hermite's formula produces expressions for x_0+ph, at which $f(x)$ is known, in terms of pivotal values of f and f' at neighbouring points x_0+ih, where i ranges from $-[\frac{1}{2}(n-1)]$ to $[\frac{1}{2}n]$. Results are given for $n=2(1)7$, where n is the number of points required in direct Hermite interpolation.

The expressions are somewhat complicated. For desk-machine work, in particular, some subtabulation, followed by easier methods of inverse interpolation, would appear to be generally quicker and less liable to blunders.

L. Fox (Teddington).

Stelson, Hugh E. Note on using the reciprocal function for a linear inverse interpolation. Math. Tables Aids Comput. **10** (1956), 32–34.

Ostrowski, Alexander. Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen. Comment. Math. Helv. **30** (1956), 175–210.

The linear iterations treated are generalizations of the cyclic single step method of solving a linear algebraic equation system [see Ostrowski Rend. Mat. e Appl. (5) **14** (1954), 140–163; MR **16**, 1155]. Such iterations are called "absolutely convergent" for a matrix A, if the iteration remains convergent (with fixed choices of the variable or variables being relaxed) when the elements of A are multiplied by arbitrary factors of absolute value 1. Let $\alpha_{\mu\mu}=|a_{\mu\mu}|$. Let the "companion matrix to A" be the matrix $A_B=[(2\delta_{\mu\nu}-1)\alpha_{\mu\nu}$. A is called an "H-matrix" if simultaneously all $\alpha_{\mu\mu}>0$, $\det A_B>0$, and all principal minors of A_B are non-negative. It is proved that A is an H-matrix if and only if the cyclic single step iteration converges absolutely, and also if and only if the entire-step (Jacobi) process converges absolutely. Estimates of the convergence rate are given, as are the possible limits of over- and underrelaxation. Group relaxation and variable "steering" are also treated. There are several other results.

This is a comprehensive report with 10 theorems, proofs, and 37 references. The H matrices were treated in an earlier paper [Ostrowski, Comment. Math. Helv. **10** (1937), 69–96], for which certain corrections are given on p. 187.

G. E. Forsythe (Los Angeles, Calif.).

Guest, J. The solution of linear simultaneous equations by matrix iteration. Austral. J. Phys. **8** (1955), 425–439.

One of the n-step methods of conjugate gradients [Hestenes and Stiefel, J. Res. Nat. Bur. Standards **49** (1952), 409–436; MR **15**, 651] is modified very substantially to apply to the solution of a system (1) $Du+l=0$, where $D\neq D^*$ (star denotes transpose). In the corresponding Hestenes-Stiefel procedure for non-symmetric D the relaxations are made in a set of D^*D-conjugate directions, and the successive residuals are orthogonal. However, following a method of Lanczos [ibid. **45** (1950), 255–282; MR **13**, 163], the author generates two sets of D-biconjugate directions, and relaxes along them while solving simultaneously (1) and an extraneous system (2) $D^*v+m=0$. The n successive residuals of (1) and those of (2) then form a biorthogonal system. As a biproduct, the iteration parameters can yield the characteristic polynomial of D. Several properties of the method are developed, and numerical examples of orders 3 and 6 are exhibited.

Since the eigenvalues of D^*D are more widely spaced than those of D, the author expects and seems to find more accurate solutions with his method than with Stiefel's, although it takes slightly more computation. Matrix inversion, checking and round-off errors are discussed.

G. E. Forsythe (Los Angeles, Calif.).

Marcus, M. On the optimum gradient method for systems of linear equations. Proc. Amer. Math. Soc. **7** (1956), 77–81.

The optimum gradient method for solving a system of linear equations $Ax=b$ proceeds [see A. S. Householder, Principles of numerical analysis, McGraw-Hill, New York, 1953; MR **15**, 470] by step-by-step minimization of the quadratic form $(Ax-b)^*R(Ax-b)$, where R is some positive definite hermitean matrix. At each step one gains at least a factor $\eta_R=1-4(p+p^{-1})^{-2}$, where $p=(\lambda_M/\lambda_m)^{\frac{1}{2}}$, and λ_M, λ_m are the maximal and minimal eigenvalues of A^*RA. The author assumes that A itself is positive definite hermitean, and takes $R=\psi^2(A)$, where $\psi(x)$ is such that $x^2\psi^2(x)$ increases and $\psi^2(x)$ decreases in the interval (a, b) containing the eigenvalues of A. It is shown that $\eta_R\leqq\eta_I$, where I is the unit matrix. Actually the necessary and sufficient condition on ψ is that (a, b) is mapped by $|x\psi(x)|$ into an interval (c, d) with $c^{-1}d\leqq a^{-1}b$.

N. G. de Bruijn (Amsterdam).

Bandyopadhyay, G.; and Narasimhan, R. K. Special types of group relaxation for simultaneous linear equations. Quart. J. Mech. Appl. Math. **9** (1956), 122–128.

The process is illustrated for three equations in three

unknowns and the process for this case summarized as follows: "(i) movement to the surface of any one of the planes; (ii) movement to the line of intersection of this plane with any other plane and still remaining on the first plane; (iii) movement to the common point of intersection of all the planes by remaining on the line reached in process (ii)." *A. S. Householder* (Oak Ridge, Tenn.).

Lelderman, Yu. R. On a method of solution of simultaneous algebraic linear equations. Akad. Nauk Üzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 153–158. (Russian)

The author is considering methods for accelerating the convergence of convergent linear iterative processes for solving nth order systems of linear algebraic equations. If vectors x_p ($p=1, 2, \cdots$) approaching the solution are found by iteration, let $\Delta_p=x_p-x_{p-1}$. The author observes that, for p so large that only k eigenvalues of the iteration process effectively remain, the Δ_p practically satisfy a linear difference equation of order $k\leq n$:

$$(*) \qquad B_0\Delta_p+B_1\Delta_{p+1}+\cdots+B_k\Delta_{p+k}=0 \ (B_k\neq 0).$$

But (*) implies that

$$\sum_{j=p}^{\infty}\Delta_j=[B_1\Delta_p+B_2(\Delta_p+\Delta_{p+1})+\cdots +B_{k-1}(\Delta_p+\cdots+\Delta_{p+k-1})]\times[B_0+\cdots+B_k]^{-1}.$$

From this formula one can find $\lim x_p$ at once. It is noted that the B_i in (*) can be computed from one component of the Δ_p.

The cases $k=1$, 2 are used as illustrations. The case $k=1$ was derived by L. Lyusternik [Trudy Mat. Inst. Steklov. **20** (1947), 49–64; MR **10**, 71].

There are several misprints, and the reader must deduce the definitions of certain letters. See the two following reviews for papers preceding this one. *G. E. Forsythe.*

Lelderman, Yu. R. On a method of solution of a system of linear algebraic equations when the ordinary methods of successive approximation are inapplicable. Dokl. Akad. Nauk Üzbek. SSR. **1953**, no. 1, 8–11. (Russian. Uzbek summary)

Let x solve the system $x=\lambda Ax+f$ of simultaneous linear algebraic equations, where $0<\lambda_1<\lambda_2<\cdots<\lambda_n$ are the numbers for which $u=\lambda Au$ has a nontrivial solution u. The ordinary linear iteration $x_p=\lambda Ax_{p-1}+f$ ($p=1, 2, \cdots$) is known to converge to x for arbitrary x_0 if and only if $|\lambda|<\lambda_1$. The problem is how to get a convergent iteration for $\lambda_1<\lambda<\lambda_2$.

To solve the problem, the author rederives as new the δ^2 acceleration process. [Reviewer's note: for references dating back to Kummer (1837) see p. 309 of Forsythe, Bull. Amer. Math. Soc. **59** (1953), 299–329; MR **15**, 65; and S. Lubkin, J. Res. Nat. Bur. Standards **48** (1952), 228–254; MR **14**, 500; and Agnew's review in MR **14**, 500.] In this the author forms a sequence $\{\bar{x}_p\}$, defined by

$$(*) \qquad \bar{x}_p=(x_{p+1}{}^2-x_{p+2}x_p)(x_p-2x_{p+1}+x_{p+2})^{-1};$$

presumably the author interprets (*) separately for each component.

It is shown that $\bar{x}_p\to x$ for $\lambda_1<\lambda<\lambda_2$ (true for $\lambda_1<|\lambda|<\lambda_2$ — reviewer). To speed the convergence of $\bar{x}_p$ to x the author derives the case $k=1$ of the acceleration of the preceding review.

There are too many misprints to list.

G. E. Forsythe (Los Angeles, Calif.).

Lelderman, Yu. R. On solution of a system of linear algebraic equations by the method of successive approximations. Dokl. Akad. Nauk Üzbek. SSR **1953**, no. 10, 6–9. (Russian. Uzbek summary)

In the notation of the last two reviews, the object is to solve $x=\lambda Ax+f$. The author derives the formula for $\sum_{j=p}^{\infty}\Delta_j$ of the second preceding review for $k=2$. He also shows how to apply the δ^2 process [here attributed to Aitken — see the preceding review] to find λ_1. He defines $U_p=\Delta_{p+1}/\Delta_p$, presumably interpreted componentwise. Then he asserts that

$$(U_pU_{p+2}-U_{p+1}{}^2)(U_p-2U_{p+1}+U_{p+2})^{-1}=\lambda_1+O(\lambda_2/\lambda_1)^{2p}.$$

G. E. Forsythe (Los Angeles, Calif.).

Narec, L. K. Computation of statically indeterminate systems by machine methods. Trudy Tallin. Politehn. Inst. Ser. A. no. **54** (1954), 52 pp. (5 plates). (Russian)

This is an exposition of elimination methods for solving systems of linear algebraic equations by pencil and by desk calculators. Specific application is made to the equations of structural mechanics. *G. E. Forsythe.*

Wittmeyer, H. Berechnung einzelner Eigenwerte eines algebraischen linearen Eigenwertproblems durch "Störiteration". Z. Angew. Math. Mech. **35** (1955), 441–452. (English, French and Russian summaries)

Let A, B be arbitrary matrices of order n. The author compares three local iterations for improving x, λ until (*) $(A-\lambda B)x=0$: (1) Wielandt's inverse (gebrochene) iteration [R. Zurmühl, Matrizen, Springer, Berlin-Göttingen-Heidelberg, 1950, pp. 311–314; MR **12**, 73]; (2) W. Kohn's method [J. Chem. Phys. **17** (1949), 670; MR **11**, 136]; (3) one form of Unger's Newton-iteration [Zurmühl, Praktische Mathematik für Ingenieure und Physiker, Springer, Berlin-Göttingen-Heidelberg, 1953, pp. 163–166; MR **15**, 470]. Then he introduces the new variational method of the title.

Suppose $x^{(0)}$, $\lambda^{(0)}$ exactly solve $(A-C-\lambda B)x=0$. The perturbed problem $(A-C+\varepsilon C-\lambda B)x=0$ reduces to (*) for $\varepsilon=1$. Let $\lambda=\lambda^{(0)}+\varepsilon\lambda^{(1)}+\cdots$; $x=x^{(0)}+\varepsilon x^{(1)}+\cdots$. (Keep $x_1=1$ for all ε.) It is shown that

$$\mu^{(1)}=y^{*\prime}Cx^{(0)}/y^{*\prime}Bx^{(0)} \quad (A-C-\lambda^{(0)}B)x^{(1)}+(C-\mu^{(1)}B)x^{(0)}=0,$$

where $(A-C-\lambda B)'y^*=0$. One takes $\lambda^{(1)}=\lambda^{(0)}+\mu^{(1)}$; $x^{(1)}=x^{(0)}+x^{(1)}$.

It is shown how to pick C, y^*, and other auxiliary quantities; generally the first row and/or column of A and B are isolated from the other elements. A specific algorithm is outlined and is said to be well suited to an automatic computer. It is necessary to solve a linear system s of type $Hy=k$ and $H'x=l$ at the ith iterative step, but H does not change with i. There is a numerical example of order 4. An operation count favors the new method slightly over (1), and considerably over (3).

G. E. Forsythe (Los Angeles, Calif.).

Bondar', N. G. On the accuracy of some approximate methods of computing the eigenvalues of square matrices. Dnepropetrovsk. Inst. Inžen. Želez.-Dorož. Transport. Trudy. **23** (1953), 61–69. (Russian)

Let A be a matrix with only positive eigenvalues, which the author calls an "oscillation matrix," although the term has a narrower denotation by Gantmaher and Krein [Oscillation matrices and kernels and small oscillations of mechanical systems, 2nd ed., Gostehizdat, Moscow-Leningrad, 1950; MR **14**, 178]. The author

quantitatively compares the precision of certain known bounds for the largest and smallest eigenvalues of A. For example, let $\mathrm{Sp}\, A^{2q}=s_2$, $\mathrm{Sp}\, A^{q}=s_1$, where Sp= "trace of." It is known that $b_1=s_2^{1/q}s_1^{-1/q}<\lambda_{\max}<s_2^{1/(2q)}=B_1$, and also that

$$b_2=[2^{-1}s_1(1+\{2s_2s_1^{-2}-1\}^{\frac{1}{2}})]^{1/q}<\lambda_{\max}<s_2^{1/(2q)}$$

[e.g., A. F. Smirnov, Static and dynamic stability of structures, Moscow-Leningrad, 1947]. The present author proves that $b_1<b_2$. He also claims to prove that $B_1-b_1<2^{1/(2q)}-1$, an incorrect result based on the incorrect assumption that $2\mathrm{Sp}\, A^2\geqq(\mathrm{Sp}\, A)^2$. [The correct result is $B_1-b_1\leqq n^{1/(2q)}-1$.] There are some numerical examples, and some discussion of the precision of these estimates when A corresponds to the vibration of certain mechanical systems. *G. E. Forsythe.*

Vyhandu, L. Generalization of Newton's method for the solution of nonlinear equations. Tartu. Gos. Univ. Trudy Estest.-Mat. Fak. **37** (1955), 114–117. (Russian. Estonian summary)

To solve a system of n algebraic equations

$$(1)\qquad y^i=f_i(x^1,\cdots,x^n)=0\quad(i=1,\cdots,n)$$

by Newton's method [e.g., L. V. Kantorovich, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380], one starts from x_0, and computes the next approximation x_1 by solving the linear system

$$(2)\ \left(\frac{\partial f_i}{\partial x^1}\right)_0(x_1^1-x_0^1)+\cdots+\left(\frac{\partial f_i}{\partial x^n}\right)_0(x_1^n-x_0^n)+f_i(x_0^1,\cdots,x_0^n)=0,$$

etc. The author proposes replacing (2) by

$$(3)\qquad a_1^i(x_1^1-x_0^1)+\cdots+a_n^i(x_1^n-x_0^n)+a_{n+1}^i=0,$$

where a_j^i are the coefficients of a certain hyperboloid of third degree having second-degree contact with (1) at $(x_0^1,\cdots,x_0^n,y_0^1)$. It is stated that this calls for

$$a_j^i=\left(\frac{\partial f_i}{\partial x_j}\right)_0-\frac{(f_i)_0}{2}\left(\frac{\partial^2 f_i}{\partial x_j^2}\right)_0\left(\frac{\partial f_j}{\partial x_j}\right)_0\ (i,j=1,\cdots,n),$$

$$a_{n+1}^i=(f_i)_0\ (i=1,\cdots,n).$$

There is no analysis of the advantages of (3) over (2). There is a numerical example for $n=2$.

G. E. Forsythe (Los Angeles, Calif.).

Lumiste, Yu. The method of steepest descent for nonlinear equations. Tartu. Gos. Univ. Trudy Estest.-Mat. Fak. **37** (1955), 106–113. (Russian. Estonian summary)

The author considers the application of the method of steepest descent in Hilbert space H [L. V. Kantorovich, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6(28), 89–185; MR **10**, 380] to finding the solution x^* of a nonlinear functional equation (1) $x+F(x)=0$. It is proved that if $F(x)=\mathrm{grad}\, f(x)$, then any x^* which extremizes $I(x)=(x,x)+2f(x)$ also solves (1). Then, for each point $x_{n-1}\in H$, define z_n as the unit vector z for which $\|2(x_{n-1}+F(x_{n-1}),z)\|$ is maximized. Define ε_n as an ε for which $(\partial/\partial\varepsilon)I(x_{n-1}+\varepsilon z_n)=0$. In the iterative process of the title one picks $x_n=x_{n-1}+\varepsilon_n z_n$.

The main theorem is that if $\|\Delta F(x)\|\leqq M\|\Delta x\|$, if $(\Delta F(x),\Delta x)\geqq m\|\Delta x\|^2$ $(m\geqq 0)$, and if $M^2<1+2m+2m^2$, then $\|x_n-x^*\|\to 0$ with the speed of a geometric progression with factor $\leqq(M^2-m^2)^{\frac{1}{2}}(1+m)^{-1}$. Moreover, x^* is unique. In practice the difficulty is to compute ε_n. Probably one can often pick $\varepsilon_n=\varepsilon$. If one picks $\varepsilon_n=-1$, one has the ordinary iteration $x_n=F(x_{n-1})$, convergent for $M<1$. If $m>0$, $(M^2-m^2)^{\frac{1}{2}}(1+m)^{-1}<M$.

A nonlinear integral equation is used as a numerical example. *G. E. Forsythe* (Los Angeles, Calif.).

Storrer, F. Amélioration du procédé de division numérique utilisant l'itération de Newton-Raphson. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 30–33.

In the iteration $y_n=y_{n-1}(2-xy_{n-1})$ for the multiplicative computation of x^{-1} the sequence y_n approaches x^{-1} monotonically from below. The author proposes to replace 2 by $2+\delta_n$, choosing δ_n so that the error segment is bisected by x^{-1} and, further, is reduced in length. Since δ_n varies with n (it must vanish in the limit) the advantage is questionable. For example, the form

$$y_n=y_{n-1}[1+(1-xy_{n-1})+(1-xy_{n-1})^2]$$

requires an additional multiplication but yields third-order convergence. The author uses a Chebyshev approach in finding δ_n and remarks that the same can be applied to reciprocal square roots. *A. S. Householder.*

Aitken, A. C. Studies in practical mathematics. VIII. On the iterative methods of Lin and Friedman for factorizing polynomials. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954–55), 190–199 (1956).

[For part VII see same Proc. **63** (1952), 326–335; MR **14**, 209.] The convergence of the methods of S. N. Lin [J. Math. Phys. **20** (1941), 231–242; MR **3**, 153] and B. Friedman [Comm. Pure Appl. Math. **2** (1949), 195–208; MR **11**, 402] for the approximation of the factors of a polynomial by iterated division is studied. The matrices which transform the errors in coefficients from one iterate to the next are explicitly found, and the criterion of convergence derived. *E. Frank* (Chicago, Ill.).

Bauer, Friedrich L. Ein direktes Iterationsverfahren zur Hurwitz-Zerlegung eines Polynoms. Arch. Elek. Übertr. **9** (1955), 285–290.

A method is described for the factorization of a real polynomial $Q(\lambda^2)$ of degree $2n$ into two real factors $E(\lambda)$, $E(-\lambda)$, where $E(\lambda)$ has no zeros in $\mathrm{Re}(\lambda)>0$ and $E(-\lambda)$ no zeros in $\mathrm{Re}\,(\lambda)<0$. The algorithm is based on the triangular decomposition of a certain infinite matrix related to the polynomial, and a linear fractional transformation of the variable. The convergence of the method is discussed. The process fulfills the requirements for computing machines. *E. Frank* (Chicago, Ill.).

Paluver, N. V. On an iterative method of factoring polynomials. Trudy Tallin. Politehn. Inst. Ser. A. no. 62 (1955), 9 pp. (Russian)

The author proposes an iterative method to find a factor $P_m(x)$ of degree m $(2\leqq m\leqq n-m)$ of a polynomial $F_n(x)$ of degree n. It is related to methods of Lin, Friedman, and Bairstow [all referenced in F. Olver, Philos. Trans. Roy. Soc. London. Ser. A. **244** (1952), 385–415; MR **14**, 209]. Here is the method when $m=2$. Let $P_2^{(0)}(x)=x^2+ax+b$ be an approximation to $P_2(x)$. Form:

$$F_n(x)=P_2^{(0)}(x)Q_2^{(0)}(x)+R_1^{(0)}(x),\ \text{where } R_1^{(0)}x=cx+d;$$

$$Q_2^{(0)}(x)=P_2^{(0)}(x)q(x)+S_1^{(0)}(x),\ \text{where } S_1^{(0)}(x)=ex+f;$$

$$R_1^{(0)}(x)=S_1^{(0)}(x)\Delta_2^{(0)}(x)+r(x)\ \text{where } r(x)=gx^3;$$

(division this last time by ascending powers of x). Let $P_2^{(1)}(x)=h[P_2^{(0)}(x)+\Delta_2^{(0)}(x)]$, where h is chosen to make

the coefficient of x^2 equal to 1. This completes one iterative step. There is a numerical example with $m=2$, $n=4$, but no convergence analysis. *G. E. Forsythe.*

Jovanović, Milan K.; and Kuzmanović, Bogdan O. A grapho-analytical method for solving algebraic equations of the fifth degree. Bull. Acad. Serbe Sci. Cl. Sci. Tech. (N.S.) **8** (1952), no. 2, 77–80.

A polynomial of fifth degree is written in the form $\eta-y\xi=0$, where $\eta=a_2x^2+a_1x+a_0$, $\xi=-(a_5x^2+a_4x+a_3)$, $y=x^3$. From the graphs of η, ξ, and y, by successive approximations a zero of the polynomial is obtained. *E. Frank* (Chicago, Ill.).

Moran, P. A. P. The numerical evaluation of a class of integrals. Proc. Cambridge Philos. Soc. **52** (1956), 230–233.

Let $f(x_1, x_2, \cdots, x_n)$ denote a multivariate normal density function whose correlation matrix is such that all its elements are equal and positive. A numerical evaluation of $\int_0^\infty \cdots \int_0^\infty f(x_1, x_2, \cdots, x_n)dx_1dx_2\cdots dx_n$ is considered. *M. Muller* (Ithaca, N.Y.).

Davis, P.; and Rabinowitz, P. Some Monte Carlo experiments in computing multiple integrals. Math. Tables Aids Comput. **10** (1956), 1–8.

The main experiments gave estimates of the volume of an n-dimensional hypersphere from samples of $N=2^\nu$ points. With pseudo-random numbers obtained by successive multiplications, estimates were given for each of $n=2, 3, \cdots, 12$ and each of $\nu=5, 6, 7, \cdots, 14$, and, in some cases, 15 and 16. For generation by Fibonacci series, results are given for $n\leqq 9$ and variable values of ν starting with $u_0=0$ and $u_1=2^{-42}$; and for $n=2$ and 9 with $u_0=\pi$, $u_1=5^{17}\cdot 2^{-42}$.

A second experiment gave estimates of $\int_0^1 x\,dx$, $\nu\leqq 18$, first using multiplicatively generated pseudo-random numbers, second using equidistributed pseudo-random numbers, vividly illustrating the advantage of the latter. Finally, a four-dimensional integral was estimated, again with equidistributed numbers.

A final observation suggests the search for best integration formulas in higher dimensions by selecting both "abscissas" and weights appropriately. *A. S. Householder* (Oak Ridge, Tenn.).

Rutishauser, Heinz. On the instability of methods for the integration of ordinary differential equations. NACA Tech. Memo. no. **1403** (1956), 15 pp.

Translation of Z. Angew. Math. Physik **3** (1952), 65–74 [MR **13**, 692].

Černyšenko, È. A. On a variant of the method of averaging. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 10–12. (Ukrainian. Russian summary)

In this variant of the method of averaging the operator of averaging has a more general form than that in the original paper on the subject by Yu. D. Sokolov [Dopovidi Akad. Nauk Ukrain. RSR **1955**, 107–111; MR **17**, 196]. *S. Kulik* (Columbia, S.C.).

Roma, Maria Sofia. Su un procedimento per il calcolo degli autovalori di problemi al contorno per equazioni differenziali di ordine $2n$. Ann. Mat. Pura Appl. (4) **40** (1955), 365–369.

Let $Q(z_k)=\sum_{i,j}^n a_{ij}z_iz_j$ be a positive semidefinite or definite form. Let $\alpha_k=\sum_{i+j=k} a_{ij}$, and let $E(u)=\sum_{k=0}^{2n}\alpha_k\partial^{2n}(u)/\partial x^k\partial y^{2n-k}$. To the region $\mathfrak{D}$ of the xy-plane associate the family $\mathfrak{D}_q$ $(0<q<1)$ of regions obtained by the affine transformation $\xi=x$, $\eta=qy$. The eigenvalue problem of the title is to solve $E(u)-(-1)^n\lambda u=0$ in $\mathfrak{D}_q$ with boundary conditions (*) $\partial^k u/\partial x^i\partial y^{k-i}=0$ $(k=0, \cdots, n-1;\ i=0, \cdots, k)$. Let $0<\lambda_1(q)<\lambda_2(q)<\cdots$ be the eigenvalues. The author sketches a variational method for getting lower bounds for the $\lambda_k(q)$, assuming that the $\lambda_k(1)$ are known. Ritz methods yield upper bounds. Let

$$\mu_q(u)=\iint_{\mathfrak{D}_q}(\partial^n u/\partial x^k\partial y^{n-k})dxdy\Big[\iint_{\mathfrak{D}_q}u^2dxdy\Big]^{-1}.$$

The author proposes using Ritz's method as formulated by Fichera [Lezioni sulle trasformazioni lineari, vol. I, Ist. Mat. Univ., Trieste, 1954, p. 234; MR **16**, 715] to find u and ϱ such that $\lambda_n(q)\leqq\mu_q(u)+\varrho\leqq\lambda_{n+1}(q)$. If this is done, the author proves that $\varrho\lambda_n(q)\geqq\varrho\mu_q(u)-[\nu_q(u)]^2+[\mu_q(u)]^2$, the main result. Numerical lower bounds from the method are tabulated without details for the fundamental eigenvalue of several elliptical clamped plates. Remark: The reader might compare the main inequality in the special case $n=1$ with that of D. H. Weinstein [Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 529–532]. *G. E. Forsythe.*

Gurevič, S. G. On application of B. G. Galerkin's method to linear problems of dynamics of systems with distributed parameters. Inžen. Sb. **22** (1955), 48–52. (Russian)

This is an expository article detailing the steps to be followed in applying Galerkin's method to the solution of a linear partial differential equation occurring in the study of dynamical systems with distributed parameters. In particular, approximate solutions are sought for the equation

$$(1)\quad a_1(x)\frac{\partial u}{\partial t}+a_2(x)\frac{\partial^2 u}{\partial t^2}+\sum_{j=1}^{h} b_j(x)\frac{\partial^j u}{\partial x^j}=\sum_{j=1}^{h} f_j(x)\psi_j(t)$$

on the interval $0\leqq x\leqq 1$, subject to the boundary conditions

$$(2)\quad \sum_{j=1}^{h-1}\left(\alpha_{jk}\frac{\partial^j u(0)}{\partial x^j}+\beta_{jk}\frac{\partial^j u(1)}{\partial x^j}\right)=\gamma_k\quad (k=1, 2, \cdots, h)$$

and the initial conditions

$$(3)\quad u(x, 0)=\omega_1(x),\ \frac{\partial u(x, 0)}{\partial t}=\omega_2(x),$$

where $a_j(x)$, $b_j(x)$, $f_j(x)$, $\omega_j(x)$, $\psi_j(t)$ are given piece-wise continuous functions on the intervals considered, and the α_{jk}, β_{jk}, γ_{jk} are known constants. A numerical example is carried out in considerable detail. *J. F. Heyda.*

Sekiya, Tsuyoshi; Saito, Atsushi; Ishimoto, Humiaki; and Tanaka, Humihiro. On the numerical calculation of the particular solution for the partial differential equation of bending of a plate. J. Japan Soc. Aero. Engrg. **1** (1953), 5–9. (Japanese. English summary)

Expressing the particular solution of

$$\Delta\Delta w=f,\ \frac{1}{8\pi}\iint \varrho^2\ln\varrho\,d\sigma,$$

approximately by $w=\sum_{X,Y} f(X, Y)W(X-x, Y-y)$, we calculated the numerical values of $W(X-x, Y-y)$. As to the partial derivatives of $w(x, y)$ up to second order, we carried out similar calculations. *Author's summary.*

★ **Poritsky, H.; and Jerrard, R. P. Graphical field plotting methods in mechanics.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 47–61. American Society of Mechanical Engineers, New York, 1955. $9.00.

Rubenstein, R. A.; Huse, Marjorie; and Machlup, Stefan. Numerical solution of the Schroedinger equation for central fields. Math. Tables Aids Comput. **10** (1956), 30–31.

Botos, Gy.; and Hosszú, M. Representation of implicit functions by pointwise nomograms. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 195–208 (1955). (Hungarian. Russian and English summaries)

This is a continuation of earlier work by J. Aczél [Acta Sci. Math. Szeged **12** (1950), Pars A, 73–80; MR **12**, 541]. The authors give necessary and sufficient conditions which an implicit function in three (respectively four) variables must satisfy in order to permit a representation by an alignment chart. The conditions have the form of partial differential equations and yield also formulae which can be used to graduate the scales. In addition to the case where all scales are on straight lines the authors consider also alignment charts for implicit functions of three variables with one curvilinear scale. *E. Lukacs.*

Kislov, B. D. Construction by means of nomograms of stability regions and of equistability margins in phase and amplitude for automatic control systems. Avtomat. i Telemeh. **16** (1955), 508–529. (Russian)

Ermilov, N. D. On nomographing the computational operations of mathematical statistics. Kirov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 7, 49–93 (18 plates). (Russian)

The construction of nomograms is discussed and a number of carefully designed alignment charts for the arithmetical operations are given. Due attention is paid to the proper selection of the scales. A number of alignment charts is constructed for some relations frequently used in elementary statistical work. *E. Lukacs.*

Smolickiĭ, H. L. Generalization of a criterion for checking the interpretation of gravity observations. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 237–238. (Russian)

Given a gravity map and a hypothetical distribution D of disturbing masses in the subsoil which is supposed to explain the observed anomalies, a relation R between the simple or/and double integrals of gravity anomaly extended to a definite area of the map and the hypothetical contours of disturbing masses is deduced. The author states that this relation R can be used to check the hypothesis D. As a theoretical relation his result is correct, but from practical point of view the variation of the density-contrast with the depth C (which is disregarded in R) will probably preclude its application. *E. Kogbetliantz.*

See also: Borisovič, p. 879; Mysovskih, p. 879; Ghosh, p. 912; Gumenyuk, 916.

Tables

Davis, P.; and Rabinowitz, P. Abscissas and weights for Gaussian quadratures of high order. J. Res. Nat. Bur. Standards **56** (1956), 35–37.

The weights a_{kn} and abscissae x_{kn}, in the Gauss quadrature formula $\int_{-1}^{1} f(x)dx=\sum_{k=1}^{n} a_{kn}f(x_{kn})$, have been computed previously, to fifteen figures, for values of n up to 16. Recent work on the N.B.S. digital computer requires similar results for larger values of n, and the weights and abscissae are given here, to 20 and 21 decimals respectively, for $n=2$, 4, 8, 16, 20, 24, 32, 40 and 48. The abscissae are the zeros of the Legendre polynomial $P_n(x)$, and were computed from Newton's rule, with the help of a first approximation involving the known zeros of $J_0(x)$. The weights were calculated from the formula $a_{kn}=2(1-x_{kn}^2)/\{nP_{n-1}(x_{kn})\}^2$. The fact that the formulae are theoretically exact for polynomials of degree not greater than $2n-1$, so that, for example, $\sum_{k=1}^{n} a_{kn}=2$ with $f(x)\equiv 1$, was used as independent check, together with the identity $\sum_{k=1}^{n/2} x_{kn}^2=-B_n/A_n$, for even n, where $P_n(x)=A_nx^n+B_nx^{n-1}+\cdots$. The discrepancy was nowhere greater than 2 units in the twentieth decimal place. *L. Fox* (Teddington).

See also: Gloden, p. 827; Salzer, p. 898; Matsunobo and Takebe, p. 924.

Mathematical Machines

Forsythe, George E. Selected references on use of high-speed computers for scientific computation. Math. Tables Aids Comput. **10** (1956), 25–27.

Bruk, I. S.; and Lenov, N. N. The electronic differential analyzer of the G. M. Kržižanovskiĭ Energetics Institute of the Academy of Sciences of the USSR. Avtomat. i Telemeh. **17** (1956), 217–227. (Russian)

Pawlak, Z. Flip-flop as generator of random binary digits. Math. Tables Aids Comput. **10** (1956), 28–30.

Johnson, D. L. Generating and testing pseudo random numbers on the IBM Type 701. Math. Tables Aids Comput. **10** (1956), 8–13.

Hunt, P. M. The electronic digital computer in aircraft structural analysis. The programming of the Argyris matrix formulation of structural theory for an electronic digital computer. I. A description of a matrix interpretive scheme and its application to a particular example. Aircraft Engrg. **28** (1956), 70–76.

★ **Wass, C. A. A. Introduction to electronic analogue computers.** McGraw-Hill Book Co., Inc., New York; Pergamon Press Ltd., London, 1955. x+237 pp. (6 plates).

This short monograph presents in an exceptionally clear and concise manner the principles of electronic analogue computation and simulation, and many of the developments and practical aspects resulting from the wide use of such equipment at the Royal Aircraft Establishment.

The simulation of several dynamical systems of varying degrees of complexity is illustrated in Chapters 2, 3, and 4, for linear and nonlinear systems, and including the simulation of a naval battle. Two methods for setting up an analyzer solution for a coupled mass-spring-friction system are described in detail in the appendix.

The backbone of the electronic computer is the high-gain d.c. amplifier. The monograph describes various amplifiers, their characteristics, their use in summing,

inverting, integrating and differentiating, and the important problem of their drift correction. Essential practical aspects, such as output limitations, input and output impedances, imperfect summers and integrators, leaky capacitors, etc., are brought into the open. For nonlinear operations, various forms of multipliers, curve followers and function generators are clearly described.

An excellent description is given of the use of simulators, introduction of initial conditions, methods of excitation, testing and adjusting the elements, time scales, and complex forms of input and feedback networks.

The final chapter describes and shows photographs of existing computers, such as the Sperry Mk.I, the GEPUS and the TRIDAC. *W. W. Soroka* (Berkeley, Calif.).

Banerji, S. K. Solution of problems by the method of analogue. Bull. Calcutta Math. Soc. **47** (1955), 1–8.

Kumar Mitra, Samarendra. Electrical analog computing machine for solving linear equations and related problems. Rev. Sci. Instrum. **26** (1955), 453–457.

For the equation $AX=G$ where A is a matrix, G a given vector, a device is set up to obtain the vector X using the Neumann series $1+\sum(1-hA)^s$, which converges to $h^{-1}A^{-1}$ for a suitable range of the parameter h. From a given $X^{(m)}$, the machine produces $X^{(m+1)}=hG+(1-hA)X^{(m)}$. This is applied in the case A is positive definite. The device produces the effect of applying a matrix B to a vector X as the voltage output of an a.c. network. The elements of $X^{(m)}$ appear as potentiometer settings and by a switching arrangement between two banks of elements are used as inputs for the computation of $X^{(m+1)}$. The parameter h is chosen on a compromise basis between convergence and accuracy requirements by a rule of thumb. Certain information relative to characteristic values is also available from the operation of the device. *F. J. Murray* (New York, N.Y.).

Burt, E. G. C.; and Lange, O. H. Function generators based on linear interpolation with applications to analogue computing. Proc. Inst. Elec. Engrs. C. **103** (1956), 51–58.

Miroux, J. Sur un nouvel analyseur analogique de régimes transitoires. Application à l'étude de certains phénomènes de régime variable. O.N.E.R.A. Publ. no. **81** (1955), 90 pp.

Liebmann, G. A new electrical analog method for the solution of transient heat-conduction problems. Trans. A.S.M.E. **78** (1956), 655–665.

Antonowicz, Kazimierz. An integrating apparatus for the Schrödinger equation. Studia Soc. Sci. Torun. Sect. A. **4** (1955), 1–24. (Polish. Russian and English summaries)

Antonowicz, K. An integrating apparatus for the Schrödinger equations. II. Acta Phys. Polon. **14** (1955), 385–393. (Russian summary)

Ugodčikov, A. G. Electromodelling of the conformal mapping of a circular cylinder onto a given doubly connected region. Ukrain. Mat. Ž. **7** (1955), 305–312. (Russian)

See also: Artobolevskiĭ, p. 910; Küchemann and Redshaw, p. 911.

ASTRONOMY

Ertel, Hans. Hydrostatische Homotropie im Erdinnern und Legendres Dichtegesetz. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. **1955**, no. 1, 14 pp.

Befindet sich flüssige Materie im hydrostatischen Gleichgewicht, so bestehen zwischen dem Potential Φ, dem Druck p und der (als stetig vorausgesetzten) Dichte ϱ "Homotropie-Relationen" der Gestalt (1) $H_1(\Phi, \varrho)=0$ und (2) $H_2(p, \varrho)=0$, die, falls ϱ dem kugelsymmetrischen Gesetz $\varrho=\varrho_0 \sin(\mu r)/\mu r$ (Legendre) genügt, die Gestalt

$$(3)\quad \Phi+\frac{4\pi\varkappa}{\mu^2}\varrho=\text{const. und } (4)\quad p-\frac{2\pi\varkappa}{\mu^2}\varrho^2=\text{const.}$$

annehmen ($\varkappa$ Gravitationskonstante, μ=const.). Es fragt sich, ob auch andere (nicht kugelsymmetrische) Dichtefelder auf die Relationen (3) und (4) führen können. Dies ist in der Tat der Fall. Verf. zeigt nämlich, daß unter der einfachen Annahme

$$h_\Phi^\varrho \equiv -\frac{\partial H_1}{\partial \Phi}\Big/\frac{\partial H_1}{\partial \varrho}=\text{const}<0$$

die Homotropie-Relationen (1) und (2) auch jetzt noch von der Form (3) und (4) sein müssen; hieraus läßt sich die Existenz von Dichtefeldern folgern, die keine Kugelsymmetrie besitzen und doch den Relationen (3) und (4) genügen. Wird eine konstante Rotationsgeschwindigkeit ω angenommen, so sind das diejenigen Dichteverteilungen, die durch Lösungen der (zeitfreien) Wellengleichung $\Delta\varrho^*+\mu^2\varrho^*=0$ ($\varrho^*=\varrho-\omega/2\pi\varkappa$) dargestellt werden. Für $\omega=0$ ist darin das Legendresche Dichtegesetz enthalten. *K. Maruhn* (Dresden).

Hide, Raymond. The character of the equilibrium of a heavy, viscous, incompressible, rotating fluid of variable density. I. General theory. Quart. J. Mech. Appl. Math. **9** (1956), 22–34.

This paper discusses the problem described in the title by essentially the same methods that were employed by Rayleigh [Proc. London Math. Soc. **14** (1883), 170–177] for the case of an inviscid, non-rotating fluid. The extension of Rayleigh's work to include the effects of viscosity was given by Chandrasekhar [Proc. Cambridge Philos. Soc. **51** (1955), 162–178; MR **16**, 639] while Bjerknes, Bjerknes, Solberg, and Bergeron [Physikalische Hydrodynamik, Springer, Berlin, 1933] extended a particular case studied by Rayleigh to include the effects of rotation. The present paper takes simultaneous account of the effects of viscosity and rotation.

D. Layzer (Cambridge, Mass.).

Hide, Raymond. The character of the equilibrium of a heavy, viscous, incompressible, rotating fluid of variable density. II. Two special cases. Quart. J. Mech. Appl. Math. **9** (1956), 35–50.

This paper treats two special cases of the theory developed in the paper reviewed above. The two cases are those discussed by Rayleigh: an exponential density law, and superposed fluids of different (constant) density. The axis of rotation is taken to be parallel to the gradient of the density.

D. Layzer (Cambridge, Mass.).

Cowling, T. G. Magneto-hydrodynamic oscillations of a rotating fluid globe. Proc. Roy. Soc. London. Ser. A. 233 (1955), 319–322.

Walén [On the vibratory rotation of the sun, Lindståhl, Stockholm, 1949] and, independently, Plumpton and Ferraro [Astrophys. J. 121 (1955), 168–174; MR 17, 199] have studied the small-amplitude torsional oscillations of a perfectly conducting fluid sphere pervaded by a poloidal magnetic field. These authors showed that each shell generated by rotating a magnetic line of force round the axis of symmetry vibrates independently of all other such shells. Since the parameters determining the free periods of a shell vary in a continuous way from one shell to another, the sphere as a whole has a continuous spectrum of free periods corresponding to any given mode of vibration. This conclusion, however, applies only to a non-rotating sphere, since the arguments on which it is based neglect the effects of Coriolis forces. The author presents semi-qualitative arguments to show that when Coriolis forces are taken into account, the boundary conditions on the variable component of the magnetic field select a discrete set of free periods from the original continuous set. He also points out that the Coriolis forces would increase the free periods considerably in a non-gravitating fluid sphere, but not in an actual star, owing to the relatively large gravitational restoring forces produced by radial displacements. *D. Layzer.*

Agostinelli, Cataldo. Oscillazioni magneto-idrodinamiche in una massa fluida ellissoidale rotante. Influenza della viscosita. J. Math. Pures Appl. (9) 35 (1956), 7–17.

In an incompressible homogeneous conducting fluid, of density ϱ, conductivity σ and permeability μ all supposed constant, the magnetic force $\mathbf{H}$ and the velocity $\mathbf{v}$ satisfy the equations

$$\frac{\partial \mathbf{H}}{\partial t}+\operatorname{curl}(\mathbf{H}\times\mathbf{v})+\frac{c^2}{4\pi\mu\sigma}\operatorname{curl}\operatorname{curl}\mathbf{H}=0,$$

$$\frac{\partial \mathbf{v}}{\partial t}-\mathbf{v}\times\operatorname{curl}\mathbf{v}+\operatorname{grad}(\tfrac{1}{2}\mathbf{v}^2+\frac{p}{\varrho}-U)+\frac{\mu}{4\pi\varrho}\mathbf{H}\times\operatorname{curl}\mathbf{H}-\nu\Delta_2\mathbf{v}=0,$$

$$\operatorname{div}\mathbf{v}=0,\quad \operatorname{div}\mathbf{H}=0,$$

it being assumed that Maxwell's displacement current is negligible; here ν is the coefficient of kinematic viscosity and U the potential of the non-electromagnetic forces per unit mass.

There is a steady motion of a mass of such a fluid rotating uniformly about the axis of z through the mass-centre with angular velocity ω, when U is the potential of the mutual attraction of the particles of the fluid, and the magnetic field is uniform in the direction of the axis of rotation: and the free surface is then an ellipsoid of revolution. The paper is concerned with the determination of small oscillations in $\mathbf{v}$ and $\mathbf{H}$ about this steady motion, taking into account the viscosity of the fluid.

E. T. Copson (St. Andrews).

Ferraro, V. C. A. Hydromagnetic waves in a rare ionized gas and galactic magnetic fields. Proc. Roy. Soc. London. Ser. A. 233 (1955), 310–318.

This paper develops the theory of plane magneto-mechanical waves of finite amplitude in a neutral ionized gas pervaded by a uniform magnetic field at right angles to the wave front, in an approximation that neglects collisions. (In a note added in proof, the author refers to an earlier paper by E. Åström [Ark. Fys. 2 (1951), 443–457; MR 12, 778], which treats the small-amplitude case of the same problem, but in a different way.) The author points out that this approximation is appropriate for the interstellar gas, where plausible estimates of the density, temperature, and magnetic field strength, indicate that both electrons and protons spiral round the magnetic lines of force many times between collisions. (In the usual formulation of fluid electrodynamics the collision frequency is assumed to be much greater than the Larmor frequency of the electrons.)

The author shows that the waves in question exist for frequencies smaller than the Larmor frequency of the electrons. If, in addition, the frequency of such a wave is small compared with the Larmor frequency of the ions, and if the Alfvén velocity for the ionized component of the gas is small compared with the velocity of light, the wave closely resembles an Alfvén wave, except that it is of necessity circularly polarized. *D. Layzer.*

Merman, G. A. A new class of periodic solutions in the restricted problem of three bodies and in Hill's problem. Trudy Inst. Teoret. Astr. 1 (1952), 5–86. (Russian)

Since the equations of celestial mechanics do not contain the time t explicitly and they assume the time-independent energy integral, the Poincaré periodic solutions of the three-body problem contain only two arbitrary constants out of twelve constants which determine the motion. Because of the small number of arbitrary constants which are at our disposal in the Poincaré periodic solutions (and also in the Schwarzschild solutions) their applicability to astronomy is very limited.

The author constructs a new class of periodic solutions for two particular cases of the three-body problem: (i) the restricted problem of three bodies, and (ii) its limit case, the Hill problem. These new periodic solutions contain four arbitrary constants out of six constants which determine the motion.

The author's principal idea can be explained as follows. Suppose that by retaining in the equations of motion certain of the most influential terms which depend on the perturbing mass μ we have obtained a nonlinear but integrable system of equations. Take such a system of equations, instead of the linear system obtained by putting $\mu=0$, as a generating system. While the period of a periodic solution of a linear system is completely determined by its coefficients, that of a periodic solution of a nonlinear system is itself an arbitrary constant. Let the general solution of the chosen generating system be a sum of a certain number of periodic functions with arbitrary independent periods. Then every set of commensurable periods gives a periodic solution of the generating system. The corresponding solution of the originally given system of equations will contain, as in the case of the Poincaré solutions, two arbitrary constants. But, in addition, there will be as many arbitrary constants at our disposal as there are independent periods in the general solution of the nonlinear generating system. Such a nonlinear system of equations, called the averaged system of equations, has been constructed by the author for the two special cases mentioned above of the three-body problem.

A brief outline of the three chapters of the paper follows. At the beginning of Chap. I there is a theorem on the existence of periodic solutions of a non-autonomous dynamical system with n degrees of freedom which is identical to the one recently proved by the reviewer [Trans. Roy. Soc. Canada. Sect. III. (3) 49 (1955), 39–48;

MR 17, 617]. (The reviewer regrets that this paper did not come to his attention earlier!) The theorem is concerned with the case where the functional determinant of the periodicity conditions and certain equations to be satisfied by the parameters occurring in the generating solution vanish identically. Further, the author is concerned with canonical systems of equations and, in particular, with the case of identically vanishing Hessian of the Hamiltonian function in the two-body problem. This latter case, as is well-known, causes the main difficulty in applying the Poincaré theory of periodic solutions to celestial mechanics (by restricting too much the number of arbitrary constants which are at our disposal). According to the author one way of overcoming this difficulty consists in a proper choice of a generating system. For this purpose he omits, on physical grounds, all terms in the original system of equations whose mean values over the period of motion are close to zero. The averaged nonlinear system obtained in this way is then taken for the generating system and reduced to quadratures using the Jacobi theory.

In Chapters II and III the corresponding averaged systems of equations for the two special cases mentioned above are investigated in detail and the existence of new periodic solutions, containing four arbitrary constants is proved. *E. Leimanis* (Vancouver, B.C.).

Merman, G. A. On Chazy's investigations in the problem of three bodies. Byull. Inst. Teoret. Astr. **5** (1954), 594–605. (Russian)

J. Chazy [Ann. Sci. Ecole Norm. Sup. (3) **39** (1922), 29–130] showed that in the problem of three bodies for $h<0$ (h being the energy constant) and $t\to\infty$ the following four types of motion are possible: hyperbolic-elliptic, parabolic-elliptic, bounded and oscillating. There are three classes of each of the first two types of motion, depending on which of the three bodies recedes to infinity. Further, using integral invariants, Chazy [J. Math. Pures Appl. (9) **8** (1929), 353–380] arrived at the conclusion that capture in the domain $h<0$ is impossible. Recently Yu. L. Gazaryan [Moskov. Gos. Univ. Soobšč. Astr. Inst. no. **92** (1953), 23–45; MR **16**, 964] has expressed some doubts as to the correctness of certain of Chazy's deductions.

The author of the present paper proves by simplifying and carrying out in detail some of the proofs only sketched by Chazy that the probability of capture in the domain $h<0$ is zero. In particular, he shows that almost all motions which are hyperbolic-elliptic for $t\to\infty$ are also hyperbolic-elliptic for $t\to-\infty$.

Chazy claimed that these two hyperbolic-elliptic motions belong to the same class. His proof, however, is not convincing. The author of the present paper does not give a proof of this; but, by reference to a theorem of G. D. Birkhoff [Dynamical systems, Amer. Math. Soc. Colloq. Publ., vol. 9, New York, 1927 p. 282], shows that the probability that these two motions belong to the same class is different from zero. *E. Leimanis.*

Merman, G. A. A new criterion for hyperbolic motion in the problem of three bodies. Byull. Inst. Teoret. Astr. **6** (1955), 69–72. (Russian)

With reference to the three-body problem, sufficient conditions are given that all three bodies recede from each other as $t\to\infty$. This criterion is applicable if the relative velocity of some pair of the bodies is not large and the third body is at a considerable distance from that pair or recedes from it sufficiently rapidly. The author claims that, compared with all other known criteria for hyperbolic motion [G. F. Hil'mi, The problem of n bodies in celestial mechanics and cosmogony, Izdat. Akad. Nauk SSSR, Moscow, 1951; MR **17**, 417; G. A. Merman, Astr. Ž. **30** (1953), 332–339; MR **15**, 356], this criterion is preferable for the investigation of capture in the three-body problem. *E. Leimanis* (Vancouver, B.C.).

Merman, G. A. Hyperbolic approaches in the problem of three bodies. Byull. Inst. Teoret. Astr. **6** (1955), 73–84. (Russian)

G. A. Merman [same Byull. **5** (1953), 325–372, 373–391; MR **16**, 293] obtained sufficient conditions for capture in the restricted hyperbolic problem of three bodies. He replaced the actual problem by an approximate scheme, neglecting the influence of the body which recedes to infinity. Afterwards the error committed by this simplification was estimated.

In the restricted hyperbolic problem of three bodies the relative motion of two bodies is hyperbolic, and in the simplified scheme the actual motion of the third body is replaced by a rectilinear and uniform motion of this body with respect to the center of mass of the remaining two bodies. In the general problem of three bodies, in addition, the deviation of the actual relative orbit of the approaching two bodies from the hyperbola in the simplified scheme is to be estimated.

Consider the motion of three particles P_0, P_1, P_2 with the initial conditions: (i) the mutual distance r_{21} between P_1 and P_2 is small in comparison with the other two distances r_{01} and r_{02}; (ii) the relative velocities of P_1 and P_2, P_0 and P_2 are hyperbolic while the relative velocity of P_0 and P_1 is close to a parabolic one. Any motion with such initial conditions is called a close approach between P_1 and P_2. In any domain of such a close approach the receding body P_0 has little influence on the relative motion of P_1 and P_2, and the influence of the latter, in turn, is negligible on the motion of P_0 with respect to the center of mass G of P_1 and P_2. The motion of the three particles in the early stages will thus differ little from the motion of the same three particles in a simplified scheme (I) in which P_2 moves relative to P_1 in a hyperbola but the motion of P_0 relative to G is uniform and rectilinear.

The relative speed $|\dot{\mathbf{r}}_{21}|$ being great, the hyperbola described by the radius vector $\mathbf{r}_{21}$ rapidly approaches its asymptote, and the approach between P_1 and P_2 ceases to be close. As the motion goes on, the orbit described by the radius vector $\mathbf{r}_{01}$ will change its form and go over into an elliptic or a hyperbolic orbit. It is therefore assumed that in the intermediate stages the motion proceeds either according to the simplified scheme (II), in which the motion of P_0 with respect to G continues to be uniform and rectilinear but the relative orbit of $\mathbf{r}_{01}$ is a circle $r_{01}=\text{const}$, or according to the scheme (III) in which the relative motions of the three particles are all uniform and rectilinear.

In the later stages of motion, criteria which guarantee the preservation of the one or the other type of motion at infinity are applied.

The principal problem is to estimate the deviations of the true motions of the three particles from those in the three above mentioned simplified schemes. The author solves this problem by an immediate generalization to the general three-body problem of the methods developed for the same purpose in the above mentioned papers in the case of the restricted hyperbolic problem of three bodies.

This generalized method for obtaining bounds for the coordinates and the velocities of the particles for a given time interval provides at the same time sufficient conditions for a capture in the general three-body problem. *E. Leimanis* (Vancouver, B.C.).

Merman, G. A.; and Kočina, N. G. Application of a method of estimates to O. Yu. Šmidt's example. Byull. Inst. Teoret. Astr. **6** (1955), 85–111. (Russian)

The authors are concerned with the application of a method developed in the paper reviewed above to a numerical example of O. Yu. Šmidt [Dokl. Akad. Nauk SSSR (N.S.) **58** (1947), 213–216], illustrating the possibility of capture in the three-body problem. Since the conditions of capture are in the form of inequalities, the corresponding domain of the initial conditions of motion constitute an open set. That this set is not vacuous is shown in the present paper. The above mentioned paper by Merman together with the present one give the first rigorous proof that the probability of capture in the general three-body problem is positive. *E. Leimanis.*

Batrakov, Yu. V. Periodic solutions of Schwarzschild's type in the restricted problem of three bodies. Byull. Inst. Teoret. Astr. **6** (1955), 112–120. (Russian)

With reference to the circular restricted problem of three bodies in space, the existence of periodic solutions, analogous to the Schwarzschild solutions in the plane, is proved. While in the latter case the apse-line of the osculating Kepler ellipse rotates through a certain angle during the period of motion $T=T_0+\Delta T$ (T_0 being the period of the generating solution and ΔT being of the order of a small parameter), in the case of the newly obtained periodic solutions the line of nodes of the osculating ellipse is subject to a rotation. The eccentricity e and the inclination I of the generating solution are connected by a relation which has been elaborated in detail for the commensurability 3:1 of the mean motions of the asteroid and Jupiter. Terms up to the fourth degree inclusive in e and I in the expansion of the perturbation function are taken into account. Inspection of a table which represents the above mentioned relation between e and I shows that the generating solution deviates too much from the actual orbits of asteroids with commensurability 3:1. Hence the new solutions obtained seem to have little value as intermediate orbits for the commensurability case considered. *E. Leimanis.*

Batrakov, Yu. V. On periodic solutions of the third sort in the general three-body problem. Byull. Inst. Teoret. Astr. **6** (1955), 121–126. (Russian)

With reference to the general three-body problem in space, the existence of periodic solutions whose periods differ from those of the generating solutions by quantities which are of the order of a small parameter is proved. For the circular restricted three-body problem in space the existence of such a type of periodic solutions is proved in the paper reviewed above. The properties of the newly obtained periodic solutions are analogous to those of the Poincaré solutions of the third sort whose periods of motion are equal to the periods of the corresponding generating solutions. In the general three-body problem both types of the above mentioned periodic solutions are synodic; i.e., after the elapse of the period of motion only the relative positions of all three bodies are the same. *E. Leimanis* (Vancouver, B.C.).

Schubart, J. Zur Regularisierung des Zweierstosses im Dreikörperproblem. Astr. Nachr. **282** (1955), 262–264.

In the restricted three-body problem the Thiele transformation is used to regularize the equations of motion when the two finite masses have a circular motion. In the present paper the author considers the case in which the finite masses have an unperturbed motion along a conic section. The binary collision between the infinitesimal mass and one of the two finite masses is considered, and the corresponding differential equations of motion are regularized. The Sundman time transformation is used; and the two Levi-Civita vectors [Acta Math. **42** (1919), 99–144] and the osculating energy constant of the colliding masses are introduced as the three dependent variables. Since there is one superfluous dependent variable, these latter are connected by a certain relation, given in the paper. *E. Leimanis* (Vancouver, B.C.).

Mihailović, Dobrivoje. Geometric interpretation of partial gradients in the theory of planet perturbations. Bull. Soc. Math. Phys. Serbie **7** (1955), 73–84. (Serbo-Croatian. English summary)

Bucerius, H. Bahnbestimmung als Randwertproblem. V. Astr. Nachr. **282** (1955), 107–121.

Continuing previous papers [Astr. Nachr. **278** (1950), 193–203, 204–216; **280** (1951), 73–82; **281** (1953), 97–106; MR **12**, 753; **14**, 211; **15**, 749], the author makes various remarks on difficult borderline cases which arise in determining orbits of comets. He also indicates how his integral-equation method can be modified to account for perturbing forces. *W. Kaplan* (Ann Arbor, Mich.).

Onicescu, O. L'application de quelques théorèmes de calcul de probabilités à la statistique stellaire. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 8, 35–38. (Romanian. Russian and French summaries)

Let X stand for a directly unobservable characteristic of a star (e.g., absolute magnitude) and Ξ for the corresponding observable apparent characteristic of the same star (e.g., apparent magnitude) so that X is a known function $F(\Xi, R)$ of Ξ and of the star's distance R. It is assumed that X is a random variable with a probability density $\Psi(x)$, independent of R. Also it is assumed that, for a specified category of stars for which the observations of Ξ are possible, R is a random variable with a probability density $r^2D(r)$. Then the probability density of Ξ, say $\psi(\xi)$, is given by the familiar equation

$$(1) \qquad \psi(\xi)=\int_0^\infty r^2D(r)\Psi(\xi, r)(\partial\Psi/\partial\xi)dr.$$

The author notices that for three particular interpretations of X and Ξ, namely for absolute and apparent luminosity, for absolute and apparent magnitude and for transversal velocity and proper motion, there exist transformations of variables which reduce the equation (1) to a standard form of a convolution $f_1(u)=\int_{-\infty}^{\infty} f_2(v)f_3(u-v)dv$, where f_1, f_2 and f_3 are probability densities. The application of Cramér's theorem implies then that if f_1 is a Gaussian density then f_2 and f_3 are also Gaussian densities. *J. Neyman* (Berkeley, Calif.).

Roberts, P. H. The equilibrium of magnetic stars. Astrophys. J. **122** (1955), 508–512.

Ferraro [Astrophys. J. **119** (1954), 407–412; MR **16**, 183] has shown that the free surface of an axially sym-

metric, infinitely conducting, liquid star in which a poloidal magnetic field is present is, to a first approximation, and under certain simplifying assumptions about the form of the magnetic field, an oblate spheroid whose ellipticity is proportional to the ratio between the magnetic energy and the gravitational energy of the star. The present paper carries the calculation of the shape of the free surface, in the case considered by Ferraro, to terms of the third order in this ratio. *D. Layzer.*

See also: Kustaanheimo, p. 818.

RELATIVITY

★**Einstein, Albert. The meaning of relativity. 5th ed.** Princeton University Press, Princeton, N. J., 1955. vi+169 pp. $3.75.

The present (fifth) edition of this most excellent book differs from the previous ones in that it contains a new formulation of Einstein's generalized theory of relativity in Appendix II which is entitled "Relativistic theory of the non-symmetric field." This review will be wholly concerned with Appendix II, which contains a succinct thought-provoking discussion of field theories in general as well as a clear exposition of Einstein's latest field theory which is intended to describe the electromagnetic as well as the gravitational field. However, a complete physical interpretation of the various quantities entering into the field equations is not given.

The discussion begins with the definition of the "strength" of a system of field equations in terms of the (free) data which must be specified in order to obtain a particular solution of these equations. A numerical measure of the strength is proposed and evaluated for various field theories.

The role of the affine connection in a general space is next discussed as is the derivation of a curvature tensor from an asymmetric affine connection. The requirement is made that the theory involving an asymmetric affine connection Γ_{ik}^{j} and an asymmetric field tensor g_{ik} be "transposition invariant", that is, invariant under the replacement g_{ik} by $\tilde{g}_{ik}$ $(=g_{ki})$ and Γ_{ik}^{j} by $\tilde{\Gamma}_{ik}^{j}$ $(=\Gamma_{ki}^{j})$. Einstein states that he thinks that this requirement, physically speaking, corresponds to the requirement that positive and negative electricity enter symmetrically into the laws of physics. The field equations themselves are deduced from a variational principle containing an invariant Lagrangean which is a function of an asymmetric tensor g_{ij} and a quantity derived from the affine connection.

The appendix closes with a section entitled "General remarks". In this section Einstein reiterates his belief that the appearance of singularities should not be admitted in field theories and points out the necessity of postulating boundary conditions with which to supplement the field equations. However, no indication is given of what these should be. Also included in this section are some remarks concerning the role that field theory may play in the understanding of the atomistic and quantum structure of reality. Einstein calls attention to the need for mathematical work on nonlinear field theories with the statement: "··· we cannot at present compare the content of a nonlinear field theory with experience. Only a significant progress in the mathematical methods can help here."

A. H. Taub (Urbana, Ill.).

Mavridès, S. La solution générale des équations d'Einstein $g^{\mu\nu}_{+\,;\varrho}=0$. Nuovo Cimento (10) **2** (1955), 1141–1164.

The present paper by this author gives complete proofs for results which were announced in C. R. Acad. Sci. Paris **241** (1955), 173–174 [MR **17**, 305]. *M. Wyman.*

Maurer-Tison, Françoise. Sur les variétés caractéristiques de la théorie unitaire du champ d'Einstein. C. R. Acad. Sci. Paris **242** (1956), 1127–1129.

A. Lichnerowicz solved the Cauchy problem for the Einstein-Schrödinger unified field theory for the hypersurface $f(x)=0$ [Rational Mech. Anal. 3 (1954), 487–521; MR **16**, 408]. The characteristics are either

(1)a $$l^{\alpha\beta}\partial_\alpha f\partial_\beta f=0,$$

or

(2)a $$\gamma^{\alpha\beta}\partial_\alpha f\partial_\beta f=0,$$

where $\gamma^{\alpha\beta}=l^{\alpha\beta}-2K^2h^{\alpha\beta}$ $(K^2\overset{\text{def}}{=\!=}h/g)$. (The notation is the same as in the cited paper by Lichnerowicz.) The corresponding cones are

(1)b $$l_{\lambda\mu}w^\lambda w^\mu=0,$$

(2)b $$\varphi(v)\overset{\text{def}}{=\!=}[(2K^2-1)l_{\lambda\mu}+2K^2l_{\varrho\sigma}f^\varrho_\lambda f^\sigma_\mu]v^\lambda v^\mu=0$$
$$(v^\lambda\overset{\text{def}}{=\!=}l^{\lambda\mu}\partial_\mu f,\ f^\varrho_\lambda\overset{\text{def}}{=\!=}m^{\varrho\alpha}l_{\alpha\lambda})$$

respectively. Moreover,

$$u^\varrho w^\sigma l_{\varrho\sigma}=0 \text{ for } u^\varrho\overset{\text{def}}{=\!=}f^\varrho_\lambda w^\lambda$$

so that u and w are conjugate with respect to (1)b and consequently, $\varphi(w)\leqq 0$. Hence, the cone (1)b is inside of the cone C, where C is conjugate to (2)b with respect to (1)b. This suggests taking $\varrho\gamma_{\lambda\mu}$ for the metric tensor. [Remark of the reviewer: It remains to be seen if $\varrho\gamma_{\lambda\mu}$ satisfies the Schrödinger requirement of physical compatibility, Proc. Roy. Irish Acad. Sect. A. **51** (1947), 147–150; MR **10**, 311.] *V. Hlavatý.*

★**Tonnelat, Marie-Antoinette. La théorie du champ unifié d'Einstein et quelques-uns de ses développements.** Gauthier-Villars, Paris, 1955. x+156 pp. 2500 francs.

In this book the author summarizes and discusses a great body of material on the Einstein and Schrödinger unified field theories. That is, relativistic theories concerning an asymmetric tensors $g_{\mu\nu}$ and an asymmetric affine connection $\Gamma^{\varrho}_{\mu\nu}$. The previous work of the author is collected and presented in a logical coherent fashion. The results obtained by other workers are also presented and compared. Thus, in this single volume containing an introduction and seven chapters one can obtain a well written complete and succinct account of the recent work in the field.

The introduction is devoted to a brief survey of Maxwell Theory in general relativity, older unified field theories and some remarks about Einstein's later attempts at a unified field theory. Ch. I discusses the algebraic and differential formalism involved in dealing with asymmetric tensors and connections. Ch. II discusses variational principles and the derivation of the field equations obtained by Schrödinger and those obtained by Einstein. Ch. III discusses one set of the field equations, namely the set which relates the affine connection with the derivatives of the asymmetric tensor $g_{\mu\nu}$ .The author has previously discussed these equations in great detail in a

series of notes in C.R. Acad. Sci. Paris [cf. e.g., MR 13, 289; 14, 213; 16, 79, 531; 17, 201]. This chapter collects these results and presents them in a unified coherent manner. Ch. IV deals with the approximate form of the field equations obtained when various expansions are made for the symmetric and antisymmetric parts of the asymmetric tensor $g_{\mu\nu}$. Ch. V treats exact spherically symmetric solutions of the field equations. The work of Papapetrou, Wyman and Bonnor is discussed and compared among themselves. Comparison is also made with the solution of the relativistic field equations for a charged particle. The difference in the solutions is discussed. Ch. VI interprets the solutions discussed in the previous chapter in terms of the Born-Infeld theory. Ch. VII discusses the stress-energy tensor of the unified field theory, the equations of geodesics in the affine space and the problem of motion of singularities in the field.

A. H. Taub (Urbana, Ill.).

Takeno, Hyôitirô. On solutions of field equations in general relativity with an electromagnetic field. Tensor (N.S.) **5** (1955), 71–76.

Let g_{ij} be the metric tensor of the general relativity, $\mathfrak{G}$ its determinant, e_{ijkl}, E^{ijkl} the Ricci tensors ($e_{1234}=(-\mathfrak{G})^{\frac{1}{2}}$, $E^{1234}=(-\mathfrak{G})^{-\frac{1}{2}}$), F_{ij} the electromagnetic tensor and

(1) $\tilde{F}^{ij}=\frac{1}{2}E^{ijkl}F_{kl}$, $\tilde{F}^{ab}g_{ai}g_{bj}=\tilde{F}_{ij}=-\frac{1}{2}e_{ijkl}F^{kl}$

its dual tensor. Put

(2) $$f_{ij}(\alpha)=F_{ij}\cos\alpha+\tilde{F}_{ij}\sin\alpha.$$

Then it follows at once from (1) and (2) for any α

(3) $\frac{1}{4}\delta_j^i F_{kl}F^{kl}-F_{jl}F^{il}=\frac{1}{4}\delta_j^i f_{kl}(\alpha)f^{kl}(\alpha)-f_{jl}(\alpha)f^{il}(\alpha)$

(4) $e_{ijkl}f^{ij}(\pi-\alpha)f^{kl}(\pi-\alpha)=\cos 2\alpha\, e_{ijkl}F^{ij}F^{kl}$
$-2\sin 2\alpha\, F_{ij}F^{ij}.$

If the Maxwell equations hold

$$\nabla_i F^{ik}=0,\ \nabla_i\tilde{F}^{ik}=0,$$
$$\nabla_i f^{ik}(\alpha)=0,\ \nabla_i\tilde{f}^{ik}(\alpha)=0,$$

we also have

$$\frac{\partial\alpha}{\partial x^k}f^{ik}(\pi-\alpha)=0,\ \frac{\partial\alpha}{\partial x^k}\tilde{f}^{ik}(\pi-\alpha)=0,$$

and therefore according to (2)

$$\frac{\partial\alpha}{\partial x^k}F^{ik}=0,\ \frac{\partial\alpha}{\partial x^k}\tilde{F}^{ik}=0.$$

Thus for $\partial\alpha/\partial x^k\neq 0$ by virtue of (4) $F_{kl}F^{kl}=0$. Hence

(5) if $F_{kl}F^{kl}\neq 0$, then $\alpha=$const.

(3) and (5) are the main results of the paper. The author found them using another approach. *V. Hlavatý.*

Ingraham, Richard. Classical Maxwell theory with finite-particle sources. Phys. Rev. (2) **101** (1956), 1411–1419.

In previous papers [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 165–169; Nuovo Cimento (10) **1** (1955), 82–102; MR **17**, 305] the author has discussed the five-dimensional space of space-time spheres and a conformally invariant formulation of Maxwell's equations in this space where the field quantities are taken to be functions of five variables. In this paper the author discusses the solutions of this extension of the Maxwell field equations corresponding to a "particle of charge e". The solutions corresponding to continuous distributions of matter are built up from these.

The stress-energy tensor is evaluated for the solutions corresponding to a particle of charge e and the various conservations satisfied by the components of this tensor as well as by other tensors are discussed in detail. The physical interpretation of various quantities entering the theory is given. *A. H. Taub* (Urbana, Ill.).

Popovici, Andrei. Le principe de la réciprocité en théorie relativiste conforme. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 78–131. (Romanian. Russian and French summaries)

Teodorescu, Ion. Sur les équations de Maxwell dans la théorie unitaire non holonome. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 8, 43–58. (Romanian. Russian and French summaries)

Just, Kurt. Erweiterte Gravitationstheorie und Perihel-drehung. Berichtigung und Ergänzungen zu Z. Physik 140, 524 (1955). Z. Physik **144** (1956), 411–427.

The author discusses the motion of a test body in the gravitational field of another body in the theory of gravitation which allows the gravitational "constant" to vary. He proposes to represent the test body by a perfect fluid whose equations of motion are a generalized form of the usual conservation equation. By integrating over the region occupied by the body, he shows that the test body does not travel along a geodesic in the space-time whose metric tensor is determined by the other spherically symmetric body at rest. The motion of the perihelion of the orbit of the test body is studied and compared to that of the Einstein theory and that obtained by other authors. *A. H. Taub* (Urbana, Ill.).

Gheorghiță, Șt. Sur un oscillateur avec la masse variable. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1953), no. 2, 19–21. (Romanian. Russian and French summaries)

The author considers a particle which obeys the laws of special relativistic mechanics, and moves on a straight line subject to the restorting force due to a linear spring. An approximate solution of the differential equation of motion

$$m_0\ddot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-3/2}+kx=0$$

is given, and some of the physical implications of the solution are discussed. The author seems not to observe that the equation can be solved without approximations, in terms of elliptic integrals. *L. A. MacColl.*

De Simoni, Franco. Sulle equazioni di campo della teoria relativistica unitaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 297–304.

The author writes down a general variational principle for non-symmetric unified theories and deduces from it, among others, the field equations of Einstein-Straus [Einstein, Supplement to Appendix II of "The meaning of relativity, 4th ed.", Princeton 1953; MR **15**, 357], Schrödinger [Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947), 205–216 (1948); **52** (1948), 1–9; MR **9**, 310; 311], Kurşunoğlu [Phys. Rev. (2) **88** (1952), 1369–1379; MR **14**, 805], and Bonnor [Proc. Roy. Soc. London. Ser. A. **226** (1954), 366–377; MR **16**, 755]. *F. A. E. Pirani.*

Scherrer, Willy. Zur linearen Feldtheorie. IV. Statische Felder. Z. Physik **144** (1956), 373–387.

[For parts I–III see Z. Physik **139** (1954), 44–55; **140** (1955), 160–163, 164–180, 374–385; MR **16**, 635, 756; **17**, 305.] The author discusses the staticspherically symmetric solutions of the field equations of linear field theory. The latter equations are obtained from a general variational principle. A discussion is given of a variety of special cases which arise when various parameters entering into the Lagrangean are restricted in value. *A. H. Taub.*

See also: Takeno, p. 892; Norden, p. 892; Petrov, p. 892; Kursunoğlu, p. 924; Tiomno, p. 929.

MECHANICS

★**Sedov, L. I. Metody podobiya i razmernosti v mehanike. [Similarity and dimensional methods in mechanics.] 3d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 328 pp. 11.80 rubles.

The third edition of this well-known book is sixty per cent longer than the second edition [1951; MR **14**, 809]. Some of the added space is devoted to amplifying the explanations of basic ideas, especially of the relation between dimensional analysis and physical "similarity". (The author does not seem to attribute much importance to the extensions of dimensional analysis to groups other than the group $a_i' = \alpha_i a_i$.)

There are five new sections in Ch. IV, dealing with one-dimensional and radial motions of a compressible gas; some of this material has already been published. Finally, there is a new chapter on astrophysical applications. Specifically, the mass-radius and mass-luminosity laws are derived; simple solutions are given to the equilibrium equations of stellar structure; especial attention is given to novae and the propagation of shock waves through them. *G. Birkhoff* (Cambridge, Mass.).

Onicescu, O. Une mécanique nouvelle des systèmes matériels. I. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 23–33. (Romanian. Russian and French summaries)

La mécanique présentée dans ce travail prend comme point de départ les systèmes analytiques définis à l'aide de paramètres canoniques q^i, p_i $(i=1, \cdots, n)$ et d'une fonction d'état $f(q, p, t)$. (From the French summary.)

F. A. E. Pirani (London).

★**Mandelker, Jakob. The derivation of a new kinetic energy formula from the energy aspect of matter.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 135–141. American Society of Mechanical Engineers, New York, 1955. $9.00.

★**Garnier, René. Cours de cinématique. Tome II. Roulement et viration. La formule de Savary et son extension à l'espace. 3ème éd.** Gauthier-Villars, Paris, 1956. x+341 pp. 5000 francs.

This edition differs from the second [1949; MR **13**, 381] in numerous additions to the text, particularly in four appended Notes.

Manžeron, D. I. Generalization of Somov's formula on accelerations of various orders. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 705–706. (Russian)

Let D be the time-derivative when applied to a number-valued variable, and the time-derivative relative to a certain Cartesian reference frame Γ when applied to a vector. The index σ only will be used to indicate summation from 1 to 3. Let q_i $(i=1, 2, 3)$ be some generalized (not necessarily orthogonal) point coordinates, constant if the point is fixed in Γ, and $\mathbf{e}_i$ the unit vector tangent to the coordinate line i at the instantaneous position of a moving point M. The velocity vector of M (relative to Γ) is

$$\mathbf{v}=\mathbf{h}_\sigma Dq_\sigma, \quad \mathbf{h}_i=H_i\mathbf{e}_i, \tag{1}$$

whence

$$\mathbf{v}_i=\partial\mathbf{v}/\partial q_i=D\mathbf{h}_i. \tag{2}$$

From (1) and (2),

$$D\mathbf{v}=\mathbf{h}_\sigma D^2q_\sigma+\mathbf{v}_\sigma Dq_\sigma,$$

and, by induction and repeated application of (2),

$$D^n\mathbf{v}=\mathbf{h}_\sigma D^{n+1}q_\sigma+(n+1)\mathbf{v}_\sigma D^nq_\sigma+R_n \quad (n=2, 3, \cdots)$$

where R_n is a polynomial in D^mq_i, $m<n$. Hence the relation

$$\partial D^n\mathbf{v}/\partial D^nq_i=(n+1)\mathbf{v}_i, \tag{3}$$

which also holds for $n=0$, 1, if we put $D^0=1$. Let $w_i^{(n)}=\mathbf{e}_iD^n\mathbf{v}$ be the orthogonal projection of the nth acceleration of M on the tangent to the ith coordinate line at M. Then

$$H_iw_i^{(n+1)}=D(H_iw_i^{(n)})-\mathbf{v}_i\cdot D^n\mathbf{v} \quad (n=0, 1, \cdots). \tag{4}$$

If $2(n+1)T_n=(D^2\mathbf{v})^2$, (4) implies

$$H_iw_i^{(n+1)}=D(H_iw_i^{(n)})-\partial T_n/\partial D^nq_i. \tag{5}$$

Since along any trajectory the coordinate system can be chosen to make $H_i=1$, and the triad $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ orthogonal, (4) can be given the form

$$w_1^{(n+1)}=D^nw_1+\omega_2w_3^{(n)}-\omega_3w_2^{(n)}, \ ***, \tag{6}$$

where

$$\omega_1=\mathbf{e}_2\cdot D\mathbf{e}_3, \ ***,$$

the asterisks indicating cyclic permutations of the preceding equations. The paper uses more complicated expressions for the components ω_i of the angular velocity of the accompanying triad. If the latter is intrinsic, (6) coincide with formulas found by Somov in 1864 [R. Beyer, Technische Kinematik, Barth, Leipzig, 1931, p. 251]. The notations of this review, and the ranges for n, do not coincide with those of the paper. Eqs. (6) could be obtained simply from the familiar formula for the derivative of a vector in a moving rectangular triad.

A. W. Wundheiler (Chicago, Ill.).

Mangeron, D.; et Braier, A. Représentation des champs d'accélération d'ordre quelconque par des nombres complexes. Gaz. Mat. Ser. A. **7** (1955), 535–540. (Romanian. Russian and French summaries)

Manžeron, D. I. On grapho-analytic methods of the kinematics of material systems. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 897–898. (Russian)

It was shown by Somov [probably in Mém. Acad. Imp.

Sci. Pétersbourg (7) **8** (1864), no. 5] that if the nth acceleration vector of a point P_n of a rigid plane in self-coplanar motion is zero, then the projections of the nth acceleration vector of any other point Z of the plane along and across ZP_n are, respectively,

$$A_n|ZP_n|,\ B_n|ZP_n|, \tag{1}$$

where

$$A_1=\dot\theta^2,\ B_1=\ddot\theta,\ A_n=A_{n-1}+\dot\theta B_{n-1},\ B_n=B_{n-1}-\dot\theta A_{n-1},\ n>1, \tag{2}$$

$\dot\theta$ being the angular velocity of the plane. [P_n does not exist only when $\dot\theta=\ddot\theta=0$, that is, when all nth acceleration vectors are equal for every n. [Beyer (cf. the preceding review) credits the case $n=1$ to Burmester (p. 237), 1888.] In the first paper the point Z is represented by a complex number z, and the theorem is given the form

$$z^{(n)}=-(nA_n+iB_n)z+z_0^{(n)},$$

$z_0^{(n)}$ being the nth acceleration of the origin. The last formula is then used to determine the loci of P_2 for the connecting rod of a slider-crank mechanism.

The second paper gives (without proof) six theorems concerning the distribution of nth accelerations. All such theorems can, obviously, be obtained from those for first accelerations if $\dot\theta^2$ is replaced by A_n, $\ddot\theta$ by B_n, and P_1 by P_n. The theorems in the paper are either such transcriptions of the theorems attributed to Burmester, Mehmke, Bresse, and De La Hire or their corollaries [cf. Beyer, loc. cit., pp. 231–240; or Rosenauer and Willis, Kinematics of mechanisms, Sydney, 1953, pp. 134–167].

A. W. Wundheiler (Chicago, Ill.).

Bessonov, A. P.; and Dubrovskiĭ, V. A. On the paper of A. I. Taĭnov, "Kinematics of the Assur groups of second class and second type". Izv. Akad. Nauk Belorussk. SSR. **1955**, 143–146. (Russian)

The authors question the argument in a paper by Tainoŭ [Vesci Akad. Navuk Belarusk. SSR **1954**, no. 1, 183–186; MR **17**, 676], and conclude that this method is erroneous.

A. W. Wundheiler (Chicago, Ill.).

Valentine, F. A. The motion of a particle constrained to move on a rough convex curve. Amer. Math. Monthly **63** (1956), 16–20.

A particle P is constrained to move on a plane convex curve C at each of whose points the curvature exists. The coefficient of friction μ is constant. The constraint is the only force acting on P, and there is no gravity. The author proves some simple remarkable theorems on the motion of P. We quote the following. The terminal velocity with which P returns to its initial position is independent of C. Let $T(A)$ and $T'(A)$ denote the periods of time required for P to complete journeys about C in the two different directions, starting from A with a given speed, then $T(A)$ is a constant if and only if C is a circle. If $T(A)=T'(A)$ for all A, then C is a circle. *O. Bottema.*

Artobolevskiĭ, I. I. Mechanisms for drawing the pedal curves of conics. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 38–41. (Russian)

★ **Chazy, Jean. Cours de mécanique rationnelle. Tome I. Dynamique du point matériel. 4ème éd.** Gauthier-Villars, Paris, 1952. v+510 pp.

★ **Chazy, Jean. Cours de mécanique rationnelle. Tome II. Dynamique des systèmes matériels.** Réimpression de la 3ème édition. Gauthier-Villars, Paris, 1953. vi+511 pp.

Litvin-Sedoĭ, M. Z. On the equations of motion of a body separate parts of which have relative mobility of the first degree. Moskov. Gos. Univ. Uč. Zap. **172** (1954), Meh. **5**, 177–190. (Russian)

In certain cases a body composed of particles (of constant or variable mass) may be divided into a number of groups of particles such that within each group the particles are fixed relative to one another while the separate groups are allowed relative mobility.

Such a system may be constructed from $s+1$ bodies $G_0, G_1, \cdots, G_s$ each composed of a fixed number of particles. Taking one of these, say G_0, as fundamental, the others are assumed to be capable of rotation without slipping, each about an axis fixed in G_0 (in general some of these axes may coincide). In general the masses of $G_0, \cdots, G_s$ will vary (i.e. the masses of their constituent particles will vary).

If G_0 is supposed free and $G_1, \cdots, G_s$ subject to no other constraints than those indicated (so that the rotation of each of $G_1, \cdots, G_s$ is kinematically independent of those of the others) the system will have $6+s$ degrees of freedom. If to several of the G_σ ($\sigma=1, \cdots, s$) is imparted a rotation about their respective axes as a result of forces of interaction between G_σ and G_0 then each of these rotations will independently affect the subsequent motion of G_0 and hence of the system as a whole.

To determine the motion of the system under given external forces (acting on G_0) and laws of variation of mass, one requires the equations of motion of the system in generalized coordinates with a general distribution of mass, and of the axes of rotation within the system. This is the object of the present investigation. These equations may be obtained from those of a body of variable mass; all that is required is an expression for the kinetic energy. The only assumption made about the forces of interaction between $G_1, \cdots, G_s$ and G_0 is that these should be internal to the system as a whole; their physical nature is immaterial.

Thus if to each of several of the bodies $G_1, \cdots, G_s$ is imparted an angular momentum as a result only of external momenta then the resulting angular rotations can in no way affect the state of the fundamental body G_0. If, however, these rotations arise from internal forces then each rotation will have its corresponding effect on the motion of G_0 and of the system as a whole, as a consequence of the conservation of angular momentum.

The interaction of G_σ ($\sigma=1, \cdots, s$) with G_0 is specified by the behaviour of the corresponding angular coordinate $\phi_\sigma=\phi_\sigma(t)$ where t is the time. If all the ϕ_σ are specified then not only are the positions of the G_σ determined but also the (initial) conditions imposed on their angular coordinates. In this case the number of degrees of freedom of the whole system reduces to that of G_0, namely 6. Here G_0 is regarded as free, and fixed to it is a set of coordinate axes $Dxyz$. The motion of G_0 is determined by the motion of this set of axes relative to the set $OXYZ$, fixed in the inertial frame of the observer. The coordinates used are those of $D(X, Y, Z)$ and the Eulerian angles ψ, θ, γ [usually denoted by θ, ϕ, ψ].

The main bulk of the work is the determination of the kinetic energy T of the system under the most general conditions possible (for this purpose additional coordinate

systems are introduced to describe the detailed configuration and its motion). The resulting expression for T is often simplified in practice (for example as a result of different types of symmetry or collinearity of some of the axes of rotation).

Finally a special case is considered, that of two bodies G_0, G_1, both possessing symmetry about the centre of mass of G_0, taken as the centre D of the coordinate system $Dxyz$, with no external forces acting on the system, with the generalized coordinates X, Y, Z, ψ, γ cyclic and with the centre of mass D moving with constant velocity. Further specializations are considered. *R. A. Rankin.*

Gröbner, W. Über Streuungs- und Stabilisierungsfunktionen bei Differentialgleichungen der theoretischen Mechanik. Ann. Mat. Pura Appl. (4) **39** (1955), 11–14.

By adding suitable terms to the Lagrangian equations for a dynamical system, it is possible to modify the equations so that the system is stable only at discrete energy levels. This can be done in a great many different ways, and the author hints at the possibility of doing it in such a way as to provide an alternative to the quantum theory. *D. C. Lewis* (Pacific Palisades, Calif.).

★**Popov, E. P. Approximate determination of free and forced oscillations in systems of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 219–248. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

In certain physical systems such as relay systems with dead zones; systems with limited linear characteristics of saturation type; systems with dry friction, etc., free oscillations may arise. A major problem is that of determining the dependence of the frequency and amplitude of the free oscillations on the values of the various system parameters with a view toward optimum selection of the latter. Another problem is that of determining the forced oscillations of nonlinear systems having a frequency equal to the frequency of an external periodic forcing function.

These two major problems, along with that of determining the stability of the periodic solutions obtained, are treated by the author by an approximate method, the so-called method of harmonic linearization due to Krylov and Bogolyubov [Introduction to non-linear mechanics, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1937]. By assuming a periodic solution for the given nonlinear system of differential equations of the form (*) $x=a\sin\omega t$ and expanding a typical nonlinear element of the system, say $F(x, \dot{x})$, by a Fourier series through terms of the first harmonic only, the resulting harmonic linearization

$$F(x, \dot{x})=q(a, \omega)x+\frac{q_1(a, \omega)}{\omega}\dot{x},$$

$$(**)\quad q(a, \omega)=\frac{1}{\pi a}\int_0^{2\pi}F(a\sin u, a\omega\cos u)\sin u\,du,$$

$$q_1(a, \omega)=\frac{1}{\pi a}\int_0^{2\pi}F(a\sin u, a\omega\cos u)\cos u\,du,$$

is obtained. Combining (**) with the usual transfer function representation of the linear part of the system, the characteristic equation for the complete (linearized) system can then be written. Graphical techniques are then introduced for determining a and ω as functions of a given system parameter k. A similar procedure is followed in determining the amplitude and phase of forced oscillations arising from the presence of the external forcing function, $x=c\sin\omega t$ in terms of c, ω, and an arbitrary system parameter k.

Applications are given to five specific non-linear systems at the close of the article.

The harmonic linearization technique described is essentially that known to engineers in this country as "describing-function analysis". [See J. G. Truxal, Automatic feedback, control system synthesis, McGraw-Hill, New York, 1955, p. 566 et seq.] *J. F. Heyda.*

Margulies, G. Remark on kinematically preferred coordinate systems. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 152–153.

Kinematically preferred coordinate systems were introduced by T. Y. Thomas as the kinematic analog (for the study of continuous media) of geodesic coordinate systems in Riemann space. As in geodesic coordinates, the question arises: What class of kinematically preferred coordinate systems are determined by a given coordinate system and the associated point trajectory? The author shows that the preferred systems are determined to within an orthogonal transformation with constant coefficients. For the special case of rigid-body motions, the validity of this result is known. *N. Coburn* (Ann Arbor, Mich.).

See also: Sethna, p. 851; Narec, p. 899.

Fluid Mechanics, Acoustics

★**Kotschin, N. J.; Kibel, I. A.; und Rose, N. W. Theoretische Hydromechanik.** Bd. I. Akademie-Verlag, Berlin, 1954. xii+508 pp. DM 45.00.

This is a translation of Kočin, Kibel' and Roze, Teoretičeskaya Gidromehanika, Tom I, 4th ed. [Gostehizdat, Moscow, 1948]. The chapter headings are as follows. I. Die Kinematik des flüssigen Mediums. II. Die Grundgleichungen der Hydrodynamik einer idealen Flüssigkeit. III. Hydrostatik. IV. Die einfachsten Bewegungen einer idealen Flüssigkeit. V. Wirbelbewegung einer idealen Flüssigkeit. VI. Das ebene Problem der Körperbewegung in einer idealen Flüssigkeit. VII. Das räumliche Problem der Körperbewegung in einer idealen Flüssigkeit. VIII. Wellenbewegungen einer idealen Flüssigkeit.

The book has in the main the character of a textbook, though some parts (those concerned with airfoil theory and with surface waves, for example) are written in broader style. The presentation is thorough and understandable, and the book should be excellent for students. Little or no attempt is made to include more recent developments in the field; there are few references to the literature after about 1935. The reviewer found it a pleasure to read some parts of the book — particularly those dealing with specialties with which he has some familiarity. *J. J. Stoker, Jr.* (New York, N.Y.).

Küchemann, D.; and Redshaw, S. C. Some problems in aerodynamics and their solution by electrical analogy. J. Roy. Aero. Soc. **60** (1956), 191–197.

Benjamin T. B.; and Lighthill, M. J. On conoidal waves and bores. Proc. Roy. Soc. London. Ser. A. **224** (1954), 448–460.

A historical discussion of the problem (omitting

however, the systematic treatment by Keller, which covers some of the material in the present paper [Comm. Appl. Math. 1 (1948), 323–339; MR 11, 227] is followed by a rederivation of Rayleigh's theory and an interpretation of it in terms of the volume-flow rate, the energy per unit mass, and the momentum-flow rate. Various limit situations, such as the transition from a smooth bore with a wave train behind it to a bore with continuous breaking at its front, are discussed.

J. J. Stoker, Jr.

Whitham, G. B. The effects of hydraulic resistance in the dam-break problem. Proc. Roy. Soc. London. Ser. A. **227** (1955), 399–407.

In terms of the long wave, or shallow-water, theory, the motion resulting from the breaking of a dam can be treated very simply by applying the theory of characteristics to the relevant system of two first-order hyperbolic partial differential equations. This theory ignores resistance to the flow, and is obviously unrealistic near the front of the wave, especially if the wave progresses down the bed of a dry valley. The author presents a theory not subject to this defect which makes use of a resistance term varying with the square of the velocity and inversely as the depth of the stream; the basic qualitative idea is to regard the vicinity of the front of the wave as a boundary layer within which rapid changes take place. On the basis of a few reasonable assumptions a mathematical formulation is obtained which can be worked out in detail. The results are in good agreement with the earlier work of Dressler [J. Res. Nat. Bur. Standards **49** (1952), 217–225] on the same problem; however, the present theory is more readily applicable in complicated cases, and it is based on assumptions which make a more direct appeal to the physical intuition.

J. J. Stoker, Jr. (New York, N.Y.).

Power, G.; and Scott-Hutton, D. L. The slow steady motion of liquid past a semi-elliptical boss. Pacific J. Math. **5** (1955), 953–962.

The author treats the problem of a slow steady viscous flow in a plane, which is bounded at one side by a rigid plane boundary with an elliptical bulge in the center; and by a simple shear flow at large distance from the boundary. The Stokes approximation is assumed and the solution is found by mapping the half-plane into the interior of a semi-circle. As limiting cases, the solutions for a semi-circular bulge and a straight projection (major axis equal to zero) are discussed.

Y. H. Kuo (Ithaca, N.Y.).

Bader, Roger; et Gilles, Jacqueline. Sur le problème de la plaque plane semi-infinie en écoulement visqueux. Rech. Aéro. no. **49** (1956), 17–22.

This paper considers the flow past a semi-infinite flat plate, using the linearized equations for viscous compressible flow with thermal conduction neglected proposed by Lagerstrom, Cole, and Trilling [Problems in the theory of viscous compressible fluids, California Inst. Tech., 1949; MR **12**, 873]. The problem is reduced to the explicit solution of two integral equations by the Wiener-Hopf method, and an exact formula for the displacement thickness of the boundary layer is obtained. Approximate results obtained by use of an iteration procedure suggested by Lagerstrom, Cole, and Trilling are found to agree closely with the exact ones.

D. W. Dunn.

★ **Schlichting, Hermann. Boundary layer theory.** Translated by J. Kestin. McGraw-Hill, New York; Pergamon Press, London; Verlag G. Braun, Karlsruhe, 1955. xx+535 pp. $15.00.

Translation of the author's Grenzschicht-Theorie [Braun, Karlsruhe, 1951; MR **13**, 177].

★ **Yih, Chia-Shun. Stability of parallel laminar flow with a free surface.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 623–628. American Society of Mechanical Engineers, New York, 1955. $9.00.

An investigation is made of the stability with respect to wave formation of a parallel laminar flow over an inclined wall under the action of gravity. The Orr-Sommerfeld equation for the disturbance stream function is solved by a series expansion in ascending powers of the Reynolds number. Numerical calculations for the case of vertical flow show that the flow becomes unstable at very low Reynolds numbers, and that in the range of computation viscosity has a stabilizing effect.

D. W. Dunn (Baltimore, Md.).

Pai, S. I. On the stability theory of flow with finite disturbances. J. Aero. Sci. **23** (1956), 274.

Ghosh, K. M. Numerical solutions to find out spectrum function of isotropic turbulence with a fourth power law fitting at small eddy numbers. Bull. Calcutta Math. Soc. **47** (1955), 71–75.

★ **Persen, Leif N. Some remarks on von Kármán's equation of isotropic turbulence.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 667–675. American Society of Mechanical Engineers, New York, 1955. $9.00.

Reid, W. H. The skewness factor according to Obukhoff's transfer theory. J. Aero. Sci. **23** (1956), 379–380.

Krzywoblocki, M. Z. E. Bergman's linear integral operator method in the theory of compressible fluid flow. D. Review of other methods, tables. Österreich. Ing.-Arch. **10** (1956), 1–38.

In this fourth part [for part C see same Arch. **8** (1954), 237–263; MR **16**, 535] of his work on Bergman's linear operator method, the author briefly outlines the other methods used in the fluid dynamics: expansion in powers of the Mach number, expansion in powers of a thickness parameter, variational methods, etc. A review of the tables and particular formulas, list of tables, a brief outline of some examples, etc., close this part.

Author's summary.

Sedow, L. I. Theoretische Gasdynamik an der Moskauer Universität. Sowjetwissenschaft. Naturwiss. Abt. **1956**, no. 1, 77–97.

Translation of Sedov's paper in Vestnik Moskov. Univ. **10** (1955), no. 4–5, 85–99 [MR **16**, 969].

Takami, Hideo. A study of the compressible flow past a Cherry profile. J. Phys. Soc. Japan **11** (1956), 145–154.

Cet article donne l'essentiel d'un important travail consacré à la comparaison de différentes méthodes approchées proposées pour la détermination des vitesses le long d'un profil en fluide parfait compressible. Le profil choisi est un de ceux calculés par Cherry [Proc. Roy. Soc.

London. Ser. A. **196** (1949), 32–36; MR **10**, 492]; le nombre de Mach est 0,5098 et la forme est celle d'un ovale d'épaisseur relative 0,931. Bien que l'exposé ne donne pas tous les détails des calculs importants nécessités par cette étude, la rédaction est suffisante pour que le lecteur puisse se rendre compte des méthodes qui ont été utilisées. Les principales méthodes testées sont les suivantes: 1) Approximation de Kármán-Tsien, appliquée comme méthode exacte lorsque $\gamma=-1$, c'est à dire en tenant compte de la déformation du profil lorsqu'on met en relation l'écoulement étudié avec un écoulement de fluide incompressible de comparaison. On notera qu'il faut obtenir ce profil de comparaison par approximations successives, ce qui conduit à chaque stade à déterminer numériquement la représentation d'un profil sur un cercle. 2) Méthodes de développement par rapport à M^2 (Méthodes d'Imaï et de Kaplan) utilisant soit le potentiel des vitesses, soit la fonction de courant, soit de façon mixte le potentiel et la fonction de courant; les développements comprennent les termes en M^4 inclus. 3) Méthode de Meksyn [ibid. **220** (1953), 239–254; MR **15**, 264] — dont la relation avec les précédentes a été mise en évidence par Imaï —, qui réduit le calcul à la résolution d'une équation algébrique obtenue en négligeant les variations de forme des lignes de courant avec le nombre de Mach.

Les principales conclusions de l'auteur sont les suivantes: tant que le nombre de Mach critique n'est pas atteint, ces méthodes donnent une approximation convenable des résultats exacts, la méthode mixte de développement par rapport à M^2 semblant la plus efficace. Mais si l'on compare le résultat exact de Cherry avec les différents résultats approchés pour des nombres de Mach conduisant à des zones locales d'écoulement supersonique, on observe une différence importante. Par example, alors que le résultat exact prévoit un maximum de vitesse au point d'épaisseur maximum, les résultats approchés donnent en ce même point un minimum relatif. On peut mettre en cause en même temps, et la justification des méthodes approchées dans de tels domaines d'application, et la validité pratique du résultat théoriquement exact, basé sur l'existence d'un écoulement transsonique continu.

P. Germain (Paris).

Kusukawa, Ken-ichi. The application of Meksyn-Imai's method to the subsonic flow past an axisymmetric body. J. Phys. Soc. Japan **10** (1955), 1088–1092.

The theory proposed by Meksyn [Proc. Roy. Soc. London. Ser. A. **220** (1953), 239–254; MR **15**, 264] and Imai [see Takami's paper reviewed above] for subsonic plane steady flow is here extended to axisymmetric flows. The extension requires certain additional approximations. The fundamental result, i.e., a fifth-degree algebraic equation whose solution gives the ratio of compressible to incompressible local speed about a given obstacle, is the same as in plane flow. The coefficients of this expression are functions of the stream Mach number. As an example, the maximum velocity in the flow past a sphere is calculated.

W. R. Sears (Ithaca, N.Y.).

Lush, P. E.; and Cherry, T. M. The variational method in hydrodynamics. Quart. J. Mech. Appl. Math. **9** (1956), 6–21.

The authors discuss two variational principles given by Bateman for the steady irrotational isentropic motion of a compressible fluid through a region when the normal mass-flow is prescribed over the boundary. The first method depends on the velocity potential and the second on the stream function. Application is made to the subsonic streaming past a circular cylinder and the numerical results obtained are compared with those found for the same problem by other methods.

A. H. Taub.

Broer, L. J. F. Simple wave solutions of the transonic equation. J. Aero. Sci. **23** (1956), 287.

Geiringer, Hilda. Grenzlinien der Hodographentransformation. Math. Z. **63** (1956), 514–524.

L'article est consacré à l'étude des lignes limites des écoulements d'un fluide compressible, lignes qui apparaissent comme on sait lorsqu'on passe d'une solution définie dans le plan de l'hodographe à la solution correspondante définie dans le plan de l'écoulement. L'auteur retrouve par une méthode rapide les résultats classiques [e.g. R. E. Meyer, Philos. Trans. Roy. Soc. London. Ser. A. **242** (1949), 153–171; MR **11**, 473]. Le cas où le nombre de Mach est égal à 1 est l'objet d'une attention spéciale.

P. Germain (Paris).

Mahony, J. J.; and Meyer, R. E. Analytical treatment of two-dimensional supersonic flow. I. Shock-free flow. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 467–498.

Les problèmes envisagés dans ce travail concernent les écoulements supersoniques bidimensionnels et irrotationnels d'un fluide parfait compressible. L'esprit de cet important mémoire peut être défini ainsi: poser de façon claire les méthodes mathématiques adaptées à ces problèmes, étudier ces méthodes et leur mise en oeuvre approchée, les appliquer à certains types de problèmes simples. Rien n'est affirmé ici sans justification et les résultats présentés ont le mérite de permettre un examen critique très intéressant de plusieurs méthodes approchées proposées antérieurement pour l'étude de ces problèmes.

La première originalité consiste dans le choix des variables; les variables indépendantes sont les variables caractéristiques, les variables dépendantes sont des quantités U et V liées de façon simple aux composantes du gradient de vitesse le long des caractéristiques X. Ces quantités satisfont à un système linéaire canonique dont on peut ramener l'étude à celle de l'équation

$$Z_{\alpha\beta}+\varphi(\alpha-\beta)Z=0.$$

On étudie d'abord la fonction de Riemann d'une telle équation. Les auteurs donnent des développements de cette fonction de Riemann qui peuvent être considerés comme des extensions de la méthode classique de Picard. Dans les cas où ces développements seraient trop lourds, les auteurs préconisent l'obtention d'une solution du système en U, V sous forme de séries doubles. Les applications étudiées sont les suivantes: 1) réflexion d'une onde simple centrée sur une paroi plane: on montre que la solution s'obtient à partir de la fonction de Riemann sans intégration, résultat obtenu indépendemment par l'auteur de cette analyse [NACA Tech. Note no. 3299 (1955); MR **17**, 100] et on donne des indications sur l'extension de la zone d'interaction dans le plan de l'écoulement; 2) réflexion d'une onde simple non centrée sur une ligne de jet, discussion complète du jet supersonique mettant en évidence l'apparition du choc et le caractère apériodique de cet écoulement; 3) réflexion d'une onde simple sur une paroi courbe, comparaison avec les différentes théories linéaires usuelles, en particulier avec celle de Whitham [Comm. Pure Appl. Math. **5** (1952), 301–348; MR **14**, 330]. La conclusion générale que l'on peut tirer de

cette étude est que linéariser dans le plan des caractéristiques donne des résultats plus complets que de linéariser dans le plan de l'écoulement. Cette conclusion d'intérêt général avait été déjà dégagée par C. C. Lin et ses élèves. *P. Germain* (Paris).

Mahony, J. J. Analytical treatment of two-dimensional supersonic flow. II. Flow with weak shocks. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 499–515.

Cette deuxième partie concerne l'étude d'écoulements avec chocs. On suppose d'abord qu'il est légitime de négliger les variations d'entropie à travers un choc. Les variables indépendantes étant toujours les variables caractéristiques, l'auteur formule les lois du choc avec ces variables et montre comment, un choc étant donné, il représente une coupure tracée dans un écoulement continu se développant sur 3 feuillets. Ces considérations sont appliquées à l'étude théorique de 3 exemples: choc au bord d'attaque d'un profil placé dans une tuyère divergente, écoulement au voisinage du point de naissance d'un choc au sein d'un écoulement, choc formé dans le jet supersonique étudié dans la première partie. Enfin dans la dernière partie est examiné la validité des hypothèses concernant la non variation de l'entropie à travers un choc; les équations générales d'un mouvement rotationnel sont données sous une forme permettant facilement la comparaison avec les équations précédemment utilisées. On montre alors que si l'erreur faite dans la détermination du choc est bien du troisième ordre par rapport à l'intensité du choc, celle faite sur l'évaluation du gradient de vitesse de l'écoulement aval est habituellement du premier ordre. *P. Germain* (Paris).

Oguchi, Hakuro. Supersonic jet with the ambient pressure corresponding to its constant pressure point. J. Phys. Soc. Japan **11** (1956), 155–159.

Cette étude est consacré à l'étude d'un jet supersonique et parallèle débouchant dans une atmosphère à pression supérieure. On suppose l'écoulement voisin du régime critique, et les variations de pression faibles pour que l'on puisse appliquer l'approximation transsonique usuelle. A la sortie du jet apparait une onde de choc courbe faisant passer de l'écoulement supersonique à un écoulement subsonique.

Il s'agit alors de résoudre le problème aux limites suivant: ψ satisfait à une équation de Tricomi, $\psi=0$ sur l'image de l'axe BC, ψ a une valeur constante connue sur l'image de la ligne de jet AB et ψ satisfait à une relation du type "dérivée oblique" le long de l'image AC du choc, qui est un arc de polaire de choc. Outre la singularité du type source existant en B, les conditions aux limites nécessitent en général en B l'introduction d'une singularité analogue à celle que l'on rencontre dans l'écoulement autour d'un angle avec choc attaché, l'écoulement en aval du choc étant subsonique. Cette singularité disparait si les conditions de l'écoulement sont telles que A est le point de Crocco de la polaire de choc. C'est ce cas particulier qui est étudié par une méthode approchée dans laquelle ψ est représentée comme combinaison linéaire de solutions convenables, les coefficients étant déterminés de manière à satisfaire en un certain nombre de points à la condition le long de l'onde de choc. *P. Germain* (Paris).

Jones, C. W. Elements of an improved linear perturbation theory of steady supersonic flow with axial symmetry. Proc. Cambridge Philos. Soc. **52** (1956), 336–343.

L'auteur étudie les écoulements permanents de révolution d'un fluide parfait compressible; la méthode utilisée est une amélioration de la théorie des perturbations linéaires. Le potentiel des vitesses $\Phi(z, r)$ vérifie une équation aux dérivées partielles quasi linéaire; z mesure la distance le long de l'axe de révolution et r la distance à cet axe. u et v désignant les composantes de la vitesse, a la célérité du son; on pose $\Phi=U(z+\varphi)$ où φ désigne le potentiel perturbé. L'auteur établit que la fonction $\varphi(z, r)$ vérifie l'équation suivante:

$$(E) \quad (1-h^2)\varphi_{rr}+\varphi_r/r-2Mh\varphi_{rz}-k^2(M^2-1)\varphi_{zz}=0.$$

h est une constante égale à la valeur moyenne de v/a, M est le nombre de Mach constant de l'écoulement non perturbé, k est une constante telle que le produit $-(M^2-1)k^2$ soit égal à la valeur moyenne de $(1-u^2/a^2)$.

Dans une première approximation, on suppose $h=0$, $k=1$ et on obtient la solution linéariseé habituelle; à l'aide de cette solution, on calcule les valeurs constantes de h et k. L'équation (E) est alors intégreé à l'aide des fonctions hypergéométriques. La solution explicitée pour un cône dont le demi-angle au sommet a pour valeur $\theta_c=10°$ est très voisine de la solution exacte. *H. Cabannes.*

Mezencev, Romane. Sur une méthode de calcul d'un générateur d'ondes de choc fonctionnant avec un coefficient d'utilisation maximum. C. R. Acad. Sci. Paris **242** (1956), 1787–1789.

Powell, J. B. L. The effect of dihedral on the lift and drag coefficients of airfoils in supersonic flow. Quart. J. Mech. Appl. Math. **9** (1956), 51–74.

The title problem is analyzed by the method of linearized conical flows and graphical results presented for delta wings with leading edges either inside or outside of the vortex Mach cone. It is shown that the lift coefficient of a delta wing is reduced approximately by $\cos^2\alpha$ where α is the dihedral angle. The author cites a recent paper by S. Nocilla [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 177–186; MR **15**, 839] in which similar results are obtained as power series in the dihedral angle and states that the leading terms therein are in agreement with his own results but that direct comparison of the higher-order terms would be difficult. *J. W. Miles.*

Milne, R. D. The unsteady aerodynamic forces on deforming, low aspect ratio wings and slender wing-body combinations oscillating harmonically in a compressible flow. Coll. Aero. Cranfield. Rep. no. **94** (1955), 42 pp. (3 plates)

The author rederives slender body approximations for a rapidly oscillating wing [cf. H. Merbt and M. Landahl, Roy. Inst. Tech. Div. Aero. Stockholm. KTH-Aero TN 30 (1953); 31 (1954); 40 (1954)] and slowly oscillating bodies of revolution and winged bodies [cf. J. W. Miles, Aero. Quart. 2 (1950), 183–194; J. Aero. Sci. 19 (1952), 208–209; MR 12, 553] and presents some numerical results for a delta wing (on the whole, much less extensive than Merbt and Landahl), a cylindrical body with conical nose, and this same body carrying a delta wing. *J. W. Miles.*

Embleton, T. F. W. The radiation force on a spherical obstacle in a cylindrical sound field. Canad. J. Phys. **34** (1956), 276–287.

A cylindrical sound field is incident on a rigid impervious sphere which is free to oscillate. The wave equation is solved in terms of spherical coordinates based on the center of the sphere as origin. Boundary conditions are

satisfied by a suitable choice of coefficients of the scattered waves in terms of coefficients of the incident waves. Computed curves of radiation force on spheres of various sizes and as functions of distance between sphere and line source are plotted. For small distances the force is one of attraction, gradually changing to one of repulsion as the distance is increased. *W. W. Soroka* (Berkeley, Calif.).

See also: Hide, p. 903; Cowling, p. 904; Agostinelli, p. 904; Ferraro, p. 904; Margulies, p. 911; Chandrasekhar, p. 920; Hide, p. 921; Baños, p. 921; Marshall, p. 921; Itô, p. 929.

Elasticity, Plasticity

★ **Doyle, T. C.; and Ericksen, J. L. Nonlinear elasticity.** Advances in applied mechanics, vol. IV, pp. 53–115. Academic Press Inc., New York, N. Y., 1956. $10.00.

Nonlinear elasticity has undergone a striking change during the last decade, mainly due to work initiated by R. S. Rivlin. The present article "attempts to familiarize the reader with the concepts and the mathematical tools which are prerequisites for understanding current literature in this field, to correlate and assess the significance of recent investigations, and to point out areas in need of further investigation". The mathematical tools regarded by the authors as prerequisites consist mainly of general coordinate systems in Euclidean space, base vectors and tensor analysis of two point fields. The need for these particular notations is bound to be a matter of opinion. Three chapters are devoted to the definitions of deformation measures and the formulation of the general theory in which stress is expressed in terms of a strain-energy function Σ. This is followed by a discussion of the form of Σ when the material of the undeformed body is isotropic, transversely isotropic and orthotropic. A number of general solutions of special problems are mentioned in which the stress system in a body in equilibrium can be expressed in terms of a general Σ for all materials with specified structural properties, e.g. all isotropic materials or all incompressible transversely isotropic materials. Only some of the more recent general solutions are discussed in detail. Chapters IX and X contain a review of various methods used for obtaining solutions by successive approximations. The article closes with very brief accounts of the motion of surfaces in continua, and of generalizations of the theory. *A. E. Green.*

Finzi, Leo. Principio della minima energia elastica differenziale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 274–280.

It is well-known in the theory of elasticity that the principle of minimum strain energy is equivalent to the equations of equilibrium and the principle of complementary energy is equivalent to the equations of compatibility. The author discusses the corresponding relations for a continuum the mechanical behavior of which is specified by a relation between the infinitesimal increments of stress and strain that need not be linear and homogeneous. Such a differential stress-strain law is called holonomic if it can be integrated leading to a finite relation between stress and strain; otherwise the stress-strain law is called non-holonomic. (The terms "total" and "incremental" are used in the current English literature on plasticity to distinguish these two types of law.) For materials with differential stress-strain laws, principles of minimum differential energy are established. Like the corresponding finite principles of the theory of elasticity, these principles are equivalent to the equations of equilibrium or compatibility. The relation between these principles and certain variational principles of the theory of plasticity (Hill, Greenberg) is discussed. *W. Prager.*

Cristea, M. Sur les petits mouvements thermo-élastiques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 72–76. (Romanian. Russian and French summaries)

Simple and well known observations on linear thermoelasticity. *C. Truesdell* (Bloomington, Ind.).

Abramyan, B. L. On a case of the plane problem of the theory of elasticity for a rectangle. Akad. Nauk Armyan. SSR. Dokl. **21** (1955), 193–198. (Russian. Armenian summary)

The generalized plane-stress problem for a rectangle loaded symmetrically along its edges $x=\pm l$, $y=\pm h$, is solved by taking Airy's stress function in the form

$$\phi(x,y)=\sum_1^\infty \cos\alpha_k x\{A_k^{(1)}\cosh\alpha_k y+D_k^{(1)}\alpha_k y\sinh\alpha_k y\}$$
$$+\sum_1^\infty \cos\beta_k y\{A_k^{(2)}\cosh\beta_k x+D_k^{(2)}\beta_k x\sinh\beta_k x\},$$

where $\alpha_k=(2k-1)\pi/2l$, $\beta_k=(2k-1)\pi/2h$. By expanding in Fourier series, the boundary conditions are reduced to a system of equation in infinitely many unknowns of the form

$$F_m=\sum_{n=1}^\infty A_{mn}F_n+B_m,$$

where F_{2m} is a multiple of $D_m^{(1)}$, F_{2m-1} a multiple of $D_m^{(2)}$. $\sum_{n=1}^\infty |A_{mn}|$ is shown to be less than 1 and consequently the system has a unique solution.

R. C. T. Smith (Cambridge, Mass.).

★ **Novožilov, V. V. Teoriya tonkih obolочek. [Theory of thin shells.]** Gosudarstv. Izdat. Sudostroitel'noĭ Literatury, Moscow, 1951. 344 pp. 20 rubles.

This book deals with the linearized theory of thin shells, based on the two assumptions, due to G. Kirchhoff, that straight fibres perpendicular to the middle surface, remain straight and perpendicular to it after deformation and that direct stresses perpendicular to the middle surface may be neglected.

Chapter I (80 pages) gives a very lucid and detailed deduction of the basic equations from first principles. The author touches only briefly on the customary approach based on the displacement equations before introducing compatibility conditions in terms of the resultants. The principal method of solution given thus deals with six simultaneous differential equations of eighth order, the number of which may be reduced to three by introduction of complex combinations of the resultants. For infinite Poisson's ratio this reduction does not represent an approximation, while for finite Poisson's ratio it depends on the omission of certain terms of the order of the thickness of the shell divided by its radius of curvature.

Chapter II (83 pages) presents the membrane theory. After discussing loading and edge conditions, compatible with the existence of solutions, shells with rotational symmetry, subject to symmetrical and anti-symmetrical loading receive detailed attention. Fourier series are used throughout this work. For more complicated shapes, a

method of affine transformation due to F. Dishinger [Bauingenieur **17** (1936), 228–231, 260–267, 289–295] is introduced.

Chapter III (71 pages) is concerned with cylindrical shells of different lengths. Under this heading the author also includes curved plates and shells with reinforcements. A number of numerical examples, illustrating the use of the method of solution, based on the complex combinations of the resultants, are given.

Chapter IV (95 pages) deals with shells of rotational symmetry. An approximate method of integration of the differential equations is discussed in general and is then applied to a wide range of problems of a practical nature.

The author refers to 163 Russian and non-Russian publications and his general presentation excels in clarity. For the reader interested in the practical aspects of shell theory, this book should be invaluable.

J. R. M. Radok (Berkeley, Calif.).

Teodorescu, P. P. Relations entre les efforts et les déformations dans la théorie de l'équilibre des surfaces cylindriques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 61–67. (Romanian. Russian and French summaries)

For the case of cylindrical shells, the author derives a theory of bending by integrating the equations of the three-dimensional theory after linearizing the three-dimensional displacements as functions of the distance from the middle surface. [This procedure was initiated and carried out for a general shell independently by V. V. Novožilov, C. R. (Dokl. Acad. Sci.) URSS (N.S.) **38** (1943), 106–164; MR **5**, 139; R. Byrne, Univ. California Publ. Math. (N.S.) **2** (1944), 103–152; MR **5**, 250; and the reviewer, Trans. Amer. Math. Soc. **58** (1945), 96–166; MR **7**, 231.] *C. Truesdell* (Bloomington, Ind.).

★ **Naghdi, P. M.; and De Silva, C. Nevin. Deformation of elastic ellipsoidal shells of revolution.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 333–343. American Society of Mechanical Engineers, New York, 1955. $9.00.

The paper considers rotationally symmetric deformations of constant-thickness elastic ellipsoidal shells of revolution. The starting point of the work is a known system of two simultaneous linear differential equations of the second order. The system contains a large parameter and a singularity of the coefficient functions in the region of physical interest. The authors solve this system following work of Langer on asymptotic solutions. Problems considered in detail are for half of a complete ellipsoid where (i) a surface load is distributed uniformly over a small region surrounding the apex, and the edge of the shell is clamped, (ii) the edge of the shell is acted upon by radial forces and by moments applied around the edge. *E. Reissner* (Cambridge, Mass.).

Gulkanyan, N. O. On the center of bending of prismatic bars with transverse section in the form of an isosceles trapezoid and in the form of an isosceles triangle. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauk **8** (1955), no. 5, 29–39. (Russian. Armenian summary)

The author considers the bending of prisms under a load P applied in the x-direction at the center of gravity of the cross-section of the free end. For the trapezoid bounded by $x=\pm y\tan\alpha$, $y=a_1$, $y=a_2$, the boundary condition on the stress function U

$$\frac{\partial U}{\partial s}=\frac{P}{2I}[x^2-f(y)]\frac{dy}{ds}$$

reduces to $\partial U/\partial s=0$, i.e. $U=$constant, on taking $f(y)=y^2\tan^2\alpha$. The differential equation for U is solved in the form

$$U(r,\phi)=\sum f_k(r)\cos(\mu_k\phi),$$

where $\mu_k=(2k-1)\pi/2\alpha$. Thus $U(r,\pm\alpha)=0$. The boundary conditions at $y=a_1$, a_2 are satisfied approximately using Trefftz's method. This underestimates the twisting moments and consequently gives lower limits for the center of bending. Lur'e [Trudy Leningrad. Indust. Inst. **1939**, no. 3, Fiz-Mat. Nauki, Vyp. 1, 121–126] had obtained upper limits. Brief tables compare the two sets of results. *R. C. T. Smith* (Cambridge, Mass.).

Solyanik-Krassa, K. V. Bending of circular rods of variable section. Inžen. Sb. **22** (1955), 206–218. (Russian)

Saint Venant's flexure problem is formulated for a beam formed by revolving a curve in the xz-plane about the z-axis. Starting from the equilibrium equations in displacements in cylindrical coordinates, the author shows that the stress components may be expressed in terms of three functions, each satisfying the equation $(\partial^2 f/\partial r^2)-(3/r)(\partial f/\partial r)+(\partial^2 f/\partial z^2)=0$, and their derivatives with respect to r and z. The functions are determined in particular cases from the external force distribution. The following problems are solved as examples: pure flexure of a cylindrical rod (hollow or uniform), bending of a cylindrical rod (hollow or uniform) by a concentrated force, flexure of a cylindrical tube under hydrostatic pressure, pure flexure of a conical rod, bending of a cone (hollow or uniform) by a force at the vertex. *R. N. Goss.*

Šeremet'ev, M. P.; and Prusov, I. O. Determination of the pressure on a clamped bar in an opening of an infinite plate stretched in two directions. Akad. Nauk Ukrain. RSR. Prikl. Mat. **1** (1955), 378–390. (Ukrainian. Russian summary)

Gumenyuk, V. S. Computation of rectangular plates of variable thickness by the method of grids. Akad. Nauk Ukrain. RSR. Prikl. Mat. **1** (1955), 436–448. (Ukrainian. Russian summary)

Das, Sisir Chandra. On the effect of a rigid spherical inclusion in a semi-infinite elastic solid under stresses produced by a couple on the plane boundary. Z. Angew. Math. Mech. **36** (1956), 73–74.

On the basis of the equilibrium equations in terms of rotations, and with the aid of dipolar coordinates, a solution in series form is obtained for the problem described in the title of the paper. *E. Sternberg.*

Lapšin, S. G. Stress in an elastic wedge from a local tangent load. Leningrad. Inst. Inžen. Vod. Transp. Trudy **22** (1955), 193–202. (Russian)

Cohen, Hirsh; and Handelman, George. Vibrations of a rectangular plate with distributed added mass. J. Franklin Inst. **261** (1956), 319–329.

The effect on the fundamental frequency of the simultaneous addition of mass and rigidity to a portion of a uniform plate is considered. Specifically, the strip

$a-l \leq x \leq a+l$ of the plate $0 \leq x \leq 2a$, $-b \leq y \leq b$ is made rigid and a uniformly distributed mass is added to it. The sides $x=0$, $x=2a$ are simply supported while the other two sides are free. Since the stiffening tends to increase the frequency while the addition of mass tends to decrease it, the question of what the combination does is of interest. It can easily be shown that as $l \to a$, the fundamental frequency increases to infinity. Thus, the question is whether it is ever decreased.

An upper bound for the fundamental frequency is obtained by a standard Rayleigh-Ritz technique using a one-parameter family of trial functions. This upper bound is shown to be close to the known correct values for the limiting case $b=0$ (from which at least part of the trial function is derived) and is thereupon assumed to be a good approximation for all b. It is plotted for various values of the geometric and elastic constants and density ratios of the problem.

In particular, it is shown in some particular cases that the frequency decreases for sufficiently large added mass. This qualitative result, however, is quite easily derivable for arbitrary fixed values of the geometric and elastic constants by a consideration of the variational principle alone. In fact, the frequency monotonically decreases to zero when the added mass increase to infinity. Furthermore, one can show that for fixed total added mass, the frequency is monotone increasing in the strip width $2l$ and becomes less than the initial frequency as $l \to 0$. These facts may have eluded the authors because they require too much differentiability of trial functions in their variational principle. *H. F. Weinberger.*

Yuščenko, O. A. On vibration of an elastic cord with a variable mass on the end. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 529–532. (Ukrainian. Russian summary)

Tersenov, S. A. On eigenvalues and eigenfunctions of vibrations of a cylindrical shell. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 575–581. (Russian)

In a finite (x, y)-region D, whose boundary L has continuous curvature, consider the problem $l^\lambda U=0$, where $U=(u, v, w)$ and l^λ has the components

$$l_{11}=b\Delta+c\partial^2/\partial x^2+\lambda,\quad l_{22}=b\Delta+c\partial^2/\partial y^2+\lambda,$$

$$l_{12}=l_{21}=c\partial^2/\partial x\partial y,\quad l_{13}=-l_{31}=(c-b)a^{-1}\partial/\partial x,$$

$$l_{23}=-l_{32}=(c+b)a^{-1}\partial/\partial y,$$

$$l_{33}=-\tfrac{1}{3}h^2(b+c)\Delta\Delta-(b+c)a^{-2}-\tfrac{1}{3}yh^2\Delta+\lambda,$$

with the boundary conditions $u=v=w=\partial w/\partial x=\partial w/\partial y=0$. The result asserts the existence of a discrete set of eigenvalues and eigenfunctions, the latter being orthonormal and complete in a special sense. A few parts of the proof are left to the reader. Analogous results are said to hold for the case of a free boundary. *F. V. Atkinson.*

★**Tersenov, S. A. On the eigenvalues and eigenfunctions of the oscillations of cylindrical shells.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 9 pp.

Translation of the paper reviewed above.

★**Ziegler, Hans. On the concept of elastic stability.** Advances in applied mechanics, vol. IV, pp. 351–403. Academic Press Inc., New York, N. Y., 1956. $10.00.

The problems treated by the author show clearly that the correct approach to the solution of a stability problem essentially depends on the nature of the problem itself. Unless the problem is purely non-gyroscopic, the easy static approaches usually fail, and even the kinetic approach frequently is questionable, if physical imperfections are neglected. To be sure, the fact that a problem is not purely non-gyroscopic may stem from incorrect assumptions concerning the nature of the load. It is, however, only in the analysis of the system that the uncertainty of such assumptions turns up.

The examples treated, being concerned with columns and shafts, belong to a relatively narrow field of applications. The results of the principal investigations remain true, of course, for more general shapes of constructions, but since most stability problems solved in these instances have so far been concerned with comparatively simple load systems of purely non-gyroscopic type, the usual static approach has not led to serious troubles. If more eleborate loading systems will be considered, the necessity of a careful choice of the approach will become apparent.

An example which has been studied by the author but abandoned, because the static approaches did not supply a satisfactory solution, is the problem of a column loaded tangentially over its whole length. Another example, where the results obtained by the static approach are questionable in certain cases, since the load is not conservative, is the problem of a cylindrical shell subjected to torsion [for a review of this problem and its solutions see S. Timoshenko, Theory of elastic stability, McGraw-Hill, New York, 1936, p. 480]. Perhaps this is the reason for the discrepancies observed between calculated and observed values of the critical moment.

R. Gran Olsson (Trondheim).

Lovass-Nagy, V. Investigation, by means of the matrix calculus, of the stability of a beam suspended at two points. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 49–63 (1955). (Hungarian. Russian and English summaries)

The conditions for stable equilibrium against lateral buckling of a beam suspended at two points are investigated. The distributed vertical load acts along the line through the center of gravity. *G. Kron.*

Sentis, André. Sur une théorie de la visco-élasticité de la matière. C. R. Acad. Sci. Paris **241** (1955), 729–730.

The stress-strain relations and the equation of motion of the Voigt body are presented with the aid of the well-known linear operator, under the rather dubious assumption of identical relaxation times in shear and volumetric deformation. *A. M. Freudenthal* (New York, N.Y.).

Mandel, Jean. Sur les corps viscoélastiques à comportement linéaire. C. R. Acad. Sci. Paris **241** (1955), 1910–1912.

The well-known use of transforms in the establishment and solution of problems of the linear visco-elastic continuum is rediscovered. *A. M. Freudenthal.*

Sentis, André. Sur une théorie de la viscoélasticité de la matière. C. R. Acad. Sci. Paris **241** (1955), 1700–1702.

The solution is discussed of the one-dimensional equation of stationary vibration of the Voigt body.

A. M. Freudenthal (New York, N.Y.).

★**Prager, William. Probleme der Plastizitätstheorie.** Birkhäuser Verlag, Basel und Stuttgart, 1955. 100 pp. Swiss francs 12.50.

The book is based on a series of lectures which the

author gave at the Federal Polytechnic Institute (ETH) in Zurich during November and December 1954. The introductory first chapter is concerned with the mechanical behavior of plastic solids. One and two dimensional models of various kind of elastic-plastic solids are described. Some complexities of the mechanical behavior in the plastic range are studied with the aid of these models. The theories of the plastic potential and the generalized plastic potential are discussed for an elastic, perfectly plastic continuum. These theories are used to obtain the flow rules corresponding to the yield conditions of v. Mises and Tresca.

Chapter II is devoted to the mechanical behavior of structures in the plastic range. A simply indeterminate, pin-jointed truss is considered first. The bars of this truss are supposed to be elastic, perfectly plastic. It is assumed that the loads are given within a common factor that defines the intensity of loading. The plastic behavior of the truss is discussed with the aid of the geometrical representation in the stress plane in which rectangular coordinates are proportional to the intensities of loading and of a residual stress state. Extremum principles are formulated and the concepts of 'load-carrying capacity' and 'shake down' are discussed. The modifications necessary for trusses with higher degrees of redundancy or for indeterminate beams are indicated.

Chapter III is concerned with 'limit analysis' and contains methods for determining the load carrying capacity of various types of rigid, perfectly plastic structures. The fundamental theorems of limit analysis are derived and interpretations of the limit load are discussed. A portal frame is treated as a first example. The influence of axial forces is discussed. The yield condition and flow rule of plates are reviewed and problems regarding the load carrying capacities of plates are solved. The relation between this analysis and Johansen's theory is discussed. The yield conditions of cylindrical shells are explored and applied to a problem. The change of the load-carrying capacity with progressing deformation is discussed in terms of an example. The applicability of the concepts of limit analysis to the problems outside the theory of plasticity is indicated.

Chapter IV deals with finite plastic deformations. The authors geometrical treatment of the slip-line field in plastic flow is discussed in some detail. The characteristic equations of Massau, Hencky and Geiringer are established. Examples are worked out to show how the geometrical relations between the physical plane and the 'stress and hodograph planes' can be used in the graphical solution of the boundary-value problems.

The book constitutes a clear and penetrating exposition of the concepts and applications of the theory of plasticity. The author is one of the principal contributors in the field and his book provides the reader with indications of the impending developments of the theory. A very useful feature of the book is that recent achievements in the application of the theory of plasticity to the design of structures are treated from a unified point of view. This self-contained review of the structural problems will be welcome by those who are interested in the extension of the theory of the elasticity to furnish more realistic estimates of the load-carrying capacities. *E. T. Onat.*

Iljuschin, A. A. Moderne Probleme der Plastizitätstheorie. Sowjetwissenschaft. Naturwiss. Abt. **1956,** no. 1, 62–76.

Translation of a paper by Il'yušin in Vestnik Moskov. Univ. **10** (1955), no. 4–5, 101–113 [MR **16,** 976].

Panferov, V. M. A general method of solution of boundary problems in the theory of elastic-plastic deformations for the simple loading of A. A. Il'yušin. Vestnik Moskov. Univ. **1952,** no. 2, 41–62. (Russian)

The analysis of this paper stems from the classical formulae, due to C. Somigliana, for the displacements set up in an isotropic elastic body by prescribed distributions of body force and surface traction under conditions of equilibrium [see, e.g., A. E. H. Love, A treatise on the mathematical theory of elasticity, Cambridge, 4th ed., 1927, § 169]. These formulae are the integrals of the elastic displacement equations, viz.

$$\text{(A)} \qquad (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_i}\right)+\mu\nabla^2 u_i+\varrho X_i=0$$

in the usual notation, satisfying the prescribed stress boundary conditions. Now if a total-strain theory of plasticity is considered, and if attention is confined to problems of continued loading, then plastic displacement equations (formed by combining the equilibrium equations and the stress-strain equations) may be derived. These equations may be written in a form analogous to Eqs. (A) where now the "body-force term X_i" includes all the non-linear plastic displacement terms. Somigliana's formulae may be applied formally to this situation, it being supposed that the displacements are small, and thereby non-linear, integro-differential equations are obtained for the displacements. This paper deals with the case where A. A. Il'yušin's equations of plastic deformation are adopted. The main purpose is to study the form of the integro-differential equations, and this is done for the special case of a spherical mass of material. The analysis is straightforward but is lengthy. No specific applications are considered. *H. G. Hopkins* (Sevenoaks).

Panferov, V. M. On the applicability of variational methods to problems of the theory of small elastic-plastic deformations. Prikl. Mat. Meh. **16** (1952), 319–322. (Russian)

The author briefly outlines a variational analysis for the solution of problems involving the small deformations of elastic-plastic media. The procedure is based upon the Galerkin method, and incorporates the concept of total strain. *H. G. Hopkins* (Sevenoaks).

Agababyan, E. H. Dynamic extension of a hollow cylinder under conditions of ideal plasticity. Ukrain. Mat. Ž. **7** (1955), 243–252. (Russian)

Slepcova, G. P. Plastic deformation of a circular membrane under static loading. Dopovidi Akad. Nauk Ukrain. RSR **1955,** 520–524. (Ukrainian. Russian summary)

Murta, M. N. On three-dimensional plastic wave propagation. Univ. Lisboa. Revista Fac. Ci. A. (2) **4** (1955), 63–78.

The author considers the propagation of waves in a strain-hardening material that obeys the stress-strain relations proposed by Trifan [Quart. Appl. Math. **7** (1949), 201–211; MR **10,** 760]. It is shown using heuristic arguments that the wave motion is composed of dilatational and distortional waves propagating with velocities that are smaller than the respective velocities of propagation in the elastic medium which describes the stress-free behavior of the plastic material considered. *E. T. Onat.*

Rosenblyum, V. I. Approximate theory of equilibrium of plastic shells. Prikl. Mat. Meh. **18** (1954), 289–302. (Russian)

A limit analysis is made of a thin cylinder of triangular cross-section subject to internal pressure. Upper and lower bounds are obtained that are never more than 8% apart. The upper bound is based on a general expression for the rate of energy dissipation in terms of strains and curvatures of the mid-surface of the shell. This is obtained by first formulating the yield condition of v. Mises in terms of the tractions and moments (not necessarily the principal values) and then introducing a variational principle, similar to Castigliano's, that gives a relationship between the corresponding generalized total strains.

R. M. Haythornthwaite (Providence, R.I.).

Lepik, Yu. R. Once again on the cylindrical form of buckling of elastic-plastic plates. Prikl. Mat. Meh. **20** (1956), 140–143. (Russian)

A long rectangular plate is subjected to compression loading at the longer edges which are taken to be simply supported. The paper concerns the post-buckling behaviour in the plastic range. *H. G. Hopkins*

Koiter, W. T. A new general theorem on shake-down of elastic-plastic structures. Nederl. Akad. Wetensch. Proc. Ser. B. **59** (1956), 24–34.

In limit analysis of structures wide use is made of two complementary theorems that bound the load carrying capacity from above and below. In shakedown analysis, on the other hand, there had been only Melan's statical theorem [Akad. Wiss. Wien. S.-B. IIa. **147** (1938), 73–87], which furnishes lower bounds for the allowable load variations. In the present paper a complementary kinematical theorem is established, which yields upper bounds for these load variations. As no examples of the practical application of the theorem are given, it is not clear whether the author is justified in hoping that, in analogy with limit analysis, the new kinematical theorem may be more convenient to apply than Melan's statical theorem. *W. Prager* (Providence, R.I.).

Sokolovskiĭ, V. V. On stability of stratified sandy foundations. Inžen. Sb. **22** (1955), 74–82. (Russian)

The author examines the conditions of limiting equilibrium for a special non-cohesive Coulomb-type material of uniform density. In this material, the angle of internal friction is the same across all planes except the horizontal, across which it is less. In regions where slip occurs across planes other than the horizontal, stress is assumed to be proportional to distance from an origin. Using this special formulation, a solution is given for the problem of a half-space having a horizontal surface which is unloaded on one side of an origin and loaded on the other side with normal and shear tractions each proportional to the distance from the origin. *R. M. Haythornthwaite.*

Van Tuyl, A. Note on the paper 'Two-dimensional plastic stress systems with isometric principal stress trajectories'. Quart. J. Mech. Appl. Math. **7** (1954), 193.

This appendix to the paper cited in the title [Neményi and Van Tuyl, same J. **5** (1952), 1–11; MR **16**, 649] provides a shorter proof of one aspect of the problem considered there. *E. H. Lee* (Providence, R.I.).

See also: Dinnik, p. 813; Sekija, Sarto, Ishimoto and Tanaka, p. 901; Margulies, p. 911.

MATHEMATICAL PHYSICS

See also: Melvin, p. 824.

Optics, Electromagnetic Theory, Circuits

Picht, J. Über eine Methode zur systematischen Errechnung einer elektronenoptisch abbildenden Feldverteilung mit bestimmten geforderten Abbildungseigenschaften. Optik **12** (1955), 433–440.

The paper discusses the determination of a rotationally symmetrical field such that the image of a given small object perpendicular to the axis is formed at a prescribed location with a prescribed lateral magnification. The lens effect is assumed to be negligible both at the object and image locations. The method, illustrated for the case of a purely magnetic lens, consists in assuming a special form, e.g. a hyperbola, for the trajectory of a paraxial electron, the only requirement being that this trajectory be a straight line in the vicinity of both object and image and that it give the desired lateral magnification. A technique is developed for computing the field giving rise to this trajectory. As long as aberrations are ignored, the method allows a great degree of arbitrariness. A method is suggested, though not carried out, for eliminating this arbitrariness and determining fields giving the optimum image. *J. E. Rosenthal* (Passaic, N.J.).

Kas'yankov, P. P. On electron lenses whose spherical aberration is arbitrarily small. Ž. Tehn. Fiz. **25** (1955), 1639–1648. (Russian)

The electron lenses considered form static, axially symmetrical fields with the following properties: the electrostatic potential $\varphi(r, z)$ and the component of the magnetic potential $A(r, z)$ can be expanded in series in the region of the lens; $\varphi(r, z)$ does not vanish inside the lens; there is no space charge. The paper is concerned with finding a method for computing a lens satisfying these conditions and giving an arbitrarily small spherical aberration. The problem is solved for magnetic lenses by forming a sequence of magnetic fields for which the coefficient of spherical aberration tends to zero. The first terms of this series are optimal in a definite sense. It is shown that this method can also be used to form a sequence of electric fields having the same property. For those problems which admit an explicit solution for a preassigned small value of the coefficient of spherical aberration, the question is considered as to whether fields of such magnitude can be readily realized experimentally. *J. E. Rosenthal* (Passaic, N.J.).

Vandakurov, Yu. V. On some antisymmetric magnetic fields with double focussing. I. Ž. Tehn. Fiz. **25** (1955), 2545–2555. (Russian)

The antisymmetric magnetic fields discussed here are defined as being produced by north and south poles symmetric with respect to a median plane. An expression is derived for an antisymmetric magnetic field with rotational symmetry, which produces double focusing of a paraxial beam of charged particles. The nature of the field depends on the direction and the initial radius of curvature of the axial-beam trajectory which is in the median

plane. The position of the image, the magnification, the dispersion, and the aberration are calculated for certain values of these parameters. The possibility is also studied of achieving double focusing in antisymmetric fields independent of one Cartesian coordinate.

J. E. Rosenthal (Passaic, N.J.).

Saxon, David S. Tensor scattering matrix for the electromagnetic field. Phys. Rev. (2) **100** (1955), 1771–1775.

The classical scattering of an arbitrary incoming electromagnetic wave by an unrestricted scattering object, is described in terms of a tensor scattering matrix. The well known relation, by means of which the total cross section is expressed in terms of the amplitude of the imaginary part of the wave scattered in the forward direction is established using this representation. The extension of this relation to non-forward scattering is shown to be useful. Two examples of application, one in the case of backward scattering, and the other for large angle scattering from a lossless, spherically symmetric scatterer, are givne. *M. Cini* (Turin).

Ahiezer, A. I.; and Faĭnberg, Ya. B. On the interaction of coupled electromagnetic resonators with a beam of charged particles. Ž. Tehn. Fiz. **25** (1955), 2516–2525. (Russian)

It is shown that if a beam of charged particles, whose unperturbed velocity exceeds a certain critical value, moves through a system of coupled resonators, the fluctuations of charge density and velocity within the beam are propagated as waves with increasing amplitude. The electromagnetic field produced has also the form of waves with increasing amplitude. The connection is shown between the waves of increasing amplitude and the Čerenkov radiation produced when the particle is moving in a periodic structure or in a dielectric. *J. E. Rosenthal.*

Soutif-Guicherd, Jeanne. Equation des ondes circulaires dans un milieu gyroparamagnétique. C. R. Acad. Sci. Paris **242** (1956), 1418–1421.

Millar, R. F. An approximate theory of the diffraction of an electromagnetic wave by an aperture in a plane screen. Proc. Inst. Elec. Engrs. C. **103** (1956), 177–185.

Gautier, Pierre. Relation entre un champ quelconque $\mathbf{B}(r, \varphi, z)$ et le champ de révolution $\mathbf{B}^R(r, z)$ tel que $B_z^R(0, z)=B_z(0, z)$. Applications. C. R. Acad. Sci. Paris **241** (1955), 930–932.

Starting with the known formulas expressing the various electric and magnetic quantities of an arbitrary field in terms of a function U, a number of properties are derived for the axially symmetrical field (identified by the superscript R) such that $B_z^R(0, z)=B_z(0, z)$. It is shown that the various quantities of this axially symmetrical field can be derived from a function U^R which is an average of U with respect to the longitude angle φ. It is also shown that the flux of an arbitrary magnetic induction through an axially symmetrical coil is expressed in terms of the component of $\mathbf{B}$ along the axis of the coil as if the field were symmetrical with respect to that axis.

J. E. Rosenthal (Passaic, N.J.).

Durand, Emile. Expression du potentiel scalaire d'un courant linéaire par une intégrale curviligne. C. R. Acad. Sci. Paris **242** (1956), 78–81.

The initial expression for the scalar magnetic potential V^* produced at a point P by a current of intensity I in a closed loop C is $V^*=(I/4\pi)\int_S (\mathbf{n}\cdot\mathbf{r}/r^3)dS$, where S is a shell bounded by C. By introducing a vector $\mathbf{f}$ such that its curl is equal to $\mathbf{r}/r^3$, the surface integral is transformed into a line integral over the curve C. Explicit expressions are given for the case where C is a circle.

J. E. Rosenthal (Passaic, N.J.).

Hudson, G. E.; and Potts, D. H. On a class of solutions of Maxwell's electromagnetic equations. Comm. Pure Appl. Math. **9** (1956), 33–43.

The authors look for solutions of Maxwell's equations

$$\nabla\times E+\mu\frac{\partial H}{\partial t}=0,\ \nabla\times H-\varepsilon\frac{\partial E}{\partial t}=\sigma E,\ \nabla\cdot E=\nabla\cdot H=0,$$

in the form

$$E=E_0\exp\{-\alpha_0 t+i(\varphi+\omega t)\},$$
$$H=H_0\exp\{-\alpha_0 t+i(\varphi-\psi+\omega t)\},$$

where E_0, H_0, φ and ψ are real functions of position and α_0 and ω are real constants. Such solutions may be considered as extensions of the solutions representing damped monochromatic plane waves. Introducing the notation

$$k_1+il_1=\mu(\omega+i\alpha_0)\varepsilon^{-1\psi},$$
$$k_2+il_2=[\varepsilon(\omega+i\alpha_0)-i\sigma]\varepsilon^{1\psi},$$
$$E_0=|E_0|,\ H_0=|H_0|,$$

they find, that then (I) ψ is a function of φ, (II) $\nabla\varphi=\Phi(E_0\times H_0)/E_0H_0$, where $\Phi^2=|k_1k_2/(1-\psi')|$; (III) E_0, H_0 are orthogonal vectors for which $E_0^2=|k_1k_2^{-1}(1-\psi')|H_0^2$; (IV) E_0 and H_0 have the forms

$$E_0=\exp\Big\{-\int_{\varphi_0}^{\varphi}\frac{l_1}{k_1}d\varphi\Big\}\nabla\cdot\mathcal{E},$$

$$H_0=\exp\Big\{-\int_{\varphi_0}^{\varphi}\frac{l_2}{k_2}(1-\psi')d\varphi\Big\}\nabla\cdot\mathcal{H},$$

where φ_0 is an arbitrary lower limit and $\mathcal{E}$ and $\mathcal{H}$ are scalar functions of position. These conditions are necessary and sufficient. The functions $\mathcal{E}$ and $\mathcal{H}$ satisfy the equations $\nabla^2\mathcal{E}=0$ and $\nabla^2\mathcal{H}=0$. The surfaces of constant phase φ are either concentric circular cylinders, concentric spheres, or parallel planes. In the last paragraph some elementary illustrative examples are given.

H. Bremekamp (Delft).

Power, G. The Blasius formulae. Amer. Math. Monthly **62** (1955), 649–650.

A generalization of the Blasius formulae is given for the force caused by the refraction of the lines of flow at a surface of discontinuity separating different dielectric media. When there is a complex potential, which is a function of the complex variable z only, these generalized formulae reduce to the normal Blasius form. A correlation is given with problems in fluid dynamics.

J. E. Rosenthal (Passaic, N.J.).

Chandrasekhar, S. Hydromagnetic turbulence. I. A deductive theory. Proc. Roy. Soc. London. Ser. A. **233** (1955), 322–330.

This paper extends the author's theory of turbulence [same Proc. **229** (1955), 1–19; MR **16**, 968] to hydromagnetics. The basic ideas and the mathematical techniques of the present paper are exactly analogous to those of the earlier paper, in which the author was able to derive a differential equation for $Q(r, t)$, the scalar

defining the velocity covariance Q_{ij}, by introducing the assumption that the fourth-order moment $Q_{ij;kl}$ is related to the covariance as in a normal distribution. For the hydromagnetic case, the author derives a pair of coupled differential equations for $Q(r, t)$ and $H(r, t)$, the scalar defining the covariance of the magnetic field. He discusses these equations in some detail for the limiting case $t \to 0$, and vanishing viscosity and resistivity, and shows that the case of equipartition between the magnetic and the mechanical energy plays a critical role for the differential equations in question. *D. Layzer* (Cambridge, Mass.).

Chandrasekhar, S. Hydromagnetic turbulence. II. An elementary theory. Proc. Roy. Soc. London. Ser. A. 233 (1955), 330–350.

This paper extends Heisenberg's theory of turbulence to hydromagnetics. The author introduces an eddy resistivity, $\lambda(k)$, which is analogous to Heisenberg's eddy viscosity $\nu(k)$, and is related to the spectrum function $G(k)$ for the magnetic field in the same way as $\nu(k)$ is related to the spectrum function $F(k)$ for the velocity field on Heisenberg's theory. The author justifies this assumption about the eddy resistivity by an appeal to his deductive theory of hydromagnetic turbulence (see the preceding review).

Exploiting the notions of eddy viscosity and eddy resistivity, the author writes down a pair of coupled integro-differential equations for the spectrum functions $F(k)$ and $G(k)$. These equations describe the transfer and dissipation of energy, and the exchange of energy between the magnetic and the mechanical modes, and they reduce to Heisenberg's equation when $G(k) \equiv 0$. The author discusses the form of the solutions for stationary turbulence in the case of vanishing resistivity and viscosity. He finds that the solutions are of two distinct kinds, one describing a "velocity mode", the other a "magnetic mode." In both modes equipartition of energy prevails for small wave numbers; but in the velocity mode the ratio $G(k)/F(k)$ tends to zero as k tends to infinity, while in the magnetic mode the ratio tends to the value 2.6, approximately. The author conjectures that the two modes may correspond to the two ways of making the transition from finite values of resistivity and viscosity to the limit of zero resistivity and viscosity; on this view, the velocity mode would result if one first set the resistivity equal to zero and then let the viscosity decrease indefinitely, while the magnetic mode would result if one followed the opposite procedure.

The author also gives a complete numerical discussion of the solutions for the case mentioned above.

D. Layzer (Cambridge, Mass.).

Backus, G. E.; and Chandrasekhar, S. On Cowling's theorem on the impossibility of self-maintained axisymmetric homogeneous dynamos. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 105–109.

Cowling [Monthly Not. Roy. Astr. Soc. 94 (1933), 39–48] has proved the following theorem concerning dynamo action (the creation and maintenance of magnetic fields by fluid motions) under conditions of axial symmetry: No steady dynamo can exist if the magnetic lines of force and the trajectories of fluid particles are confined to meridional planes. The present paper shows that the theorem remains valid when the velocity field and the magnetic field are allowed to have toroidal components, provided that the electrical conductivity and the density are taken to be constant. The method of analysis is similar to one used by the second author in previous publications [Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 1–5; Chandrasekhar and Prendergast, ibid. 42 (1956), 5–9; MR 17, 561]. *D. Layzer* (Cambridge, Mass.).

Hide, R. Waves in a heavy, viscous, incompressible, electrically conducting fluid of variable density, in the presence of a magnetic field. Proc. Roy. Soc. London. Ser. A. 233 (1955), 376–396.

This paper applies classical methods introduced by Rayleigh to treat disturbances of small amplitude in a horizontally stratified, viscous, incompressible, electrically conducting fluid in the presence of uniform, vertical gravitational and magnetic fields. The author considers two cases in detail: (a) vanishing gravitational field, and (b) infinite electrical conductivity. In case (a), small disturbance give rise to damped magneto-mechanical oscillations, whose properties the author describes. In case (b), the author finds that the equilibrium state is stable or unstable according as the density gradient and the gravitational field are parallel or antiparallel, as is true in the absence of a magnetic field. A detailed discussion is given of the nature of the disturbances in the case of stable equilibrium. Under certain conditions, depending on the values of two dimensionless numbers that characterize the flow, an intial disturbance will always decay aperiodically. When these conditions are not fulfilled, disturbances whose wavelengths lie in a certain range will give rise to damped oscillations, while disturbances whose wavelength lie outside this range will decay aperiodically. *D. Layzer* (Cambridge, Mass.).

Baños, A., Jr. Magneto-hydrodynamic waves in incompressible and compressible fluids. Proc. Roy. Soc. London. Ser. A. 233 (1955), 350–366.

This paper discusses in considerable detail, and in a systematic way, the properties of plane waves of small amplitude in a conducting fluid pervaded by a uniform magnetic field. The author places special emphasis on a distinction between waves in which the fluid velocity is at right angles to the plane formed by the direction of the field and the wave normal, and waves in which this plane contains the fluid velocity. *D. Layzer.*

Marshall, W. The structure of magneto-hydrodynamic shock waves. Proc. Roy. Soc. London. Ser. A. 233 (1955), 367–376.

This paper deals with the detailed structure of a plane shock wave in a homogeneous, fully ionized, medium pervaded by a uniform magnetic field parallel to the shock front. Using a macroscopic description for the transfer of momentum, energy, and electric charge, but taking account of the variation of viscosity, thermal conductivity, and electrical conductivity across the shock wave, the author derives a set of three simultaneous, first-order, non-linear, differential equations of the form

$$\frac{d\nu}{d\xi} = A(\nu, \tau, \chi)$$

$$\frac{d\tau}{d\xi} = B(\nu, \tau, \chi)$$

$$\frac{d\chi}{d\xi} = C(\nu, \tau, \chi)$$

where ξ is the x-coordinate, ν is the fluid velocity, τ is the temperature, and χ is the intensity of the magnetic field

(all measured in dimensionless units). The author gives a concise, but thorough, discussion of the nature of the solutions of these equations for the special cases of high and low electrical conductivity. He finds that when the conductivity is high, the magnetic field varies as the density, and the width of the shock wave is of the same order of magnitude as it would be in the absence of a magnetic field, i.e., a few mean free paths. When the conductivity is low, two different kinds of shocks can occur, depending on whether the magnetic field lies above or below a certain critical value given by

$$\left(\frac{H_{\text{crit}}^2}{8\pi p}\right)_1 = \frac{\gamma(\gamma+1)(\eta-1)}{2(2-\gamma)\eta+2\gamma-\gamma(\gamma-1)(\eta-1)^3}\left(\eta=\frac{u_1}{u_2}\right),$$

where γ is the ratio of specific heats, H the magnetic field, p the gas pressure, u the fluid velocity, and the indices 1 and 2 refer, respectively to the regions in front of and behind the shock. When the magnetic field exceeds this critical value the shock may be much wider than a few mean free paths, the shock width being, in fact, inversely proportional to the electrical conductivity. On the other hand, when the magnetic field is smaller than the critical value, a narrow transition zone occurs, whose width is of the order of a few mean free paths, but this narrow zone is preceded by a much wider one where the magnetic field undergoes its entire variation, while the fluid velocity and the temperature undergo only a part of their total variation.

D. Layzer (Cambridge, Mass.).

Knichal, Vladimír. On Kirchhoff's laws. Mat.-Fyz. Sborník Slovensk. Akad. Vied Umení **2** (1952), no. 3–4, 13–28. (Czech. Russian summary)

The author proves once again the existence and uniqueness of the solutions in lumped impedance network theory, and considers various related combinatorial questions.

R. Bott (Princeton, N.J.).

Fekhikher, Abderrahman. Sur une expression des puissances d'une matrice et son application aux réseaux itérés. Ann. Télécommun. **10** (1955), 237–241.

Let A be a 2×2 matrix. By the Cayley-Hamilton Theorem $A^2=tA-\Delta 1$, where t is the trace of A and Δ is the determinant of A. In familiar fashion $A^n=F_n(t,\Delta)A-G_n(t,\Delta)1$, where $F_n(t,\Delta)$ is a polynomial of degree $(m-1)$ in t and $G_n(t,\Delta)$ is a polynomial of degree $(m-2)$ in t. The properties, including recurrence relations, of these polynomial are studied and applied to iterated four-poles, where (in standard manner) the matrix is that of the linear transformation which sends the input voltage and current of the four-pole into its output voltage and current.

S. Sherman (Philadelphia, Pa.).

Lovass-Nagy, V.; and Győry, T. Mathematical investigation of coupled circuits by means of the matrix calculus. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **3** (1954), 65–79 (1955). (Hungarian. Russian and English summaries)

A system of linear differential equations representing an electrical network with coupled resistors, inductors and capacitors are solved. The applied voltages are periodic functions of time.

G. Kron (Schenectady, N.Y.).

See also: Cowling, p. 904; Agostinelli, p. 904; Ferraro, p. 904.

Quantum Mechanics

Beresteckiĭ, V. B. Asymptotic behavior of electromagnetic polarization of the vacuum in the presence of meson interaction. Ž. Eksper. Teoret. Fiz. **29** (1955), 585–598. (Russian)

The author calculates the asymptotic form of the photon Green's function in quantum electrodynamics, when the charge-carrying proton field is interacting with a meson field in addition to the Maxwell field. The mesons may be either scalar or pseudoscalar, and calculations are made separately for the neutral and charge-symmetric theories. The electric charge e and the meson-proton coupling constant g are assumed to satisfy the inequalities $e^2 \ll g^2 \ll 1$. The calculations are made by the method of L. D. Landau, A. A. Abrikosov and I. M. Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 497–500, 1177–1180; MR **16**, 315, 316] who evaluated the photon Green's function in the absence of meson interactions. The details are considerably more complicated when $g\neq 0$; it is therefore remarkable that the final result for the asymptotic form of the Green's function is, in every case considered, independent of g and equal to the result obtained in the absence of meson interactions. This means that the "shape" of the proton charge-distribution at very short distances is not modified by the meson-proton interaction, a result quite contrary to naive expectations.

In his last section the author presents a simple argument, based on the renormalizability of the theory, which confirms the result of the explicit calculations.

F. J. Dyson (Princeton, N.J.).

Novožilov, Yu. V. Quantum theory of a field with causal operators and the Schwinger functional. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 47–50. (Russian)

In earlier papers [same Dokl. (N.S.) **99** (1954), 723–726; MR **16**, 778] the author constructed a formulation of field theory based on "causal operators". In this paper he derives equations which are satisfied by the field-operators in this formalism, and observes that the equations are formally identical with the functional differential equations which appear in the Green's function formalism of J. Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520].

F. J. Dyson.

Costa de Beauregard, Olivier. Le postulat d'Einstein concernant la propagation des signaux et les fonctions de Green d'Univers de la théorie quantique des champs. C. R. Acad. Sci. Paris **241** (1955), 1921–1922.

Machida, Shigeru. An exactly soluble example in quantum field theory. Progr. Theoret. Phys. **14** (1955), 407–408.

Wakita, Hitoshi. Some remarks on the applicability of the field theory from the standpoint of the distribution analysis. Progr. Theoret. Phys. **14** (1955), 260–261.

Yevick, George J.; and Percus, Jerome K. New approach to the many-body problem. Phys. Rev. (2) **101** (1956) 1186–1191.

The Fourier transforms $q(k)$ of the density of n point particles are introduced as collective coordinates in the many-body problem. The validity of the transformation from ordinary space into collective space is discussed. Conditions for the approximate equivalence of the actual

Lagrangian with one expressed in collective variables are established and lead to a preliminary discussion of the Fourier representation of a function by a finite number of terms. *J. Leite Lopes* (Rio de Janeiro).

Percus, Jerome K.; and Yevick, George J. Dynamical considerations on a new approach to the many-body problem. Phys. Rev. (2) **101** (1956), 1192–1197.

The results indicated in the paper reviewed above are further analysed. N particles in a one-dimensional box with periodic boundary conditions are replaced by N one-dimensional bounded harmonic oscillators. The equations of motion for both problems are compared and their equivalence is discussed for a potential with a repulsive core with radius of the order of one angstrom, a shallow attractive well of several angstroms and zero for higher distances. The question of the Fourier representation of a function by a finite number of terms is further examined; a suitable approximation of the δ-function is made possible by the use of gap frequencies. *J. Leite Lopes.*

Marumori, Toshio; Yukawa, Jiro; and Tanaka, Ryokichi. On the foundation of the unified nuclear model. I. Progr. Theoret. Phys. **13** (1955), 442–454.

By an intuitive, semi-empirical, argument the (shell model) Hamiltonian for a number of nucleons moving in a potential is expressed in terms of collective coordinates, which describe the shape of the nucleus, and 'individual particle' coordinates. Relations between these coordinates are introduced as subsidiary conditions on the wave function. The system differs from one proposed by A. Bohr [Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1952), no. 14] by the imposition of these conditions, and by certain velocity-dependent cross-terms between the collective and individual motions. If the latter are neglected, and the frequencies of the particle excitation are larger than those of the collective motion, the energy levels can be found, using the adiabatic approximation for the collective motion. *P. T. Matthews* (Birmingham).

Gell-Mann, M.; and Pais, A. Behavior of neutral particles under charge conjugation. Phys. Rev. (2) **97** (1955), 1387–1389.

In recent schemes for the classification of elementary particles [Gell-Mann and Pais, Proceedings of the 1954 Glasgow conference on nuclear and meson physics, pp. 342–352, Pergamon Press, London, 1955] it has been suggested, that θ^0, a neutral heavy boson which is known to decay as $\theta^0 \to \pi^+ + \pi^-$, possesses an anti-particle $\bar{\theta}^0$, distinct from itself. By an intriguing use of the principle of charge-conjugation invariance, the authors show that there must exist another decay mode for θ^0 and $\bar{\theta}^0$ (possibly $\pi^+ + \pi^- + \gamma$) with a distinctly different life time and that not more than half of all θ^0's and $\bar{\theta}^0$'s can undergo the familiar decay into two pions. *A. Salam.*

Case, K. M.; Karplus, Robert; and Yang, C. N. Strange particles and the conservation of isotopic spin. Phys. Rev. (2) **101** (1956), 874–876.

The classification schemes for the elementary particles found in recent experiments make use of an abstract space (isotopic spin space) on which one allows a continuous group of transformations which is isomorphic with the 3-dimensional rotation group. Sets of related particles are assumed to be associated with the irreducible representations of this group by rules which are dependent on their physical properties. The authors consider the consequences of replacing the continuous group by certain of its finite subgroups. More precisely, they replace it by the finite subgroups of the binary unimodular unitary-matrix covering group. The major results of this are that, (a) only a finite number of irreducible representations can occur, and (b) only a part of these are embedded in the representations of the full continuous group. The study is reduced on physical grounds to the analogs of the usual tetrahedral, octahedral, and icosahedral groups. The conclusion reached is that the representations which arise from those of the continuous group are adequate to describe present knowledge of the elementary particles. The remaining representations suggest interesting possibilities which might occur with the discovery of new particles. *E. L. Hill.*

Benedict, F. D.; Daitch, P. B.; and Breit, G. Semiclassical approximation to Coulomb excitation integrals. Phys. Rev. (2) **101** (1956), 171–174.

Gluckstern, R. L.; Lazarus, J. P.; and Breit, G. Comparison of semiclassical and quantum-mechanical Coulomb excitation integrals for large L. Phys. Rev. (2) **101** (1956), 175–177.

The quantum-mechanical cross-section involves integrals over products of radical Coulomb wave functions. In the first paper, the authors establish the connection between the so-called semi-classical approximation to the cross-section, and the approximation obtained from the quantum-mechanical cross-section by replacing the Coulomb wave functions by their JWKB approximations. They also investigate numerical examples, and conclude that within certain limits on the energy ($4<\eta<8$) the semi-classical approximation may be used with fair accuracy for the computation not only of the cross-section but also of angular correlations.

In the second paper the agreement between quantum-mechanical and semi-classical computations is shown analytically for very small energies ($\eta \to 0$), provided the energy used in the semi-classical calculation is the geometric mean of the initial and final energies, and $L+\frac{1}{2}$ is used as the semi-classical quantum number. *A. Erdélyi.*

Born, Max. Continuity, determinism, and reality. Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 2, 26 pp.

The author's thesis is that it is misleading to compare quantum mechanics with deterministically formulated classical mechanics. Instead one should first reformulate classical theory, even for a single particle, in a statistical manner, in terms of a time-dependent probability density $P(x, v, t)$ for the coordinate x and velocity v at the instant t. This is because for most classical systems all orbits are unstable and, the future state can be predicted only if the initial state is defined absolutely sharply, in the sense of the mathematical concept of a point in a continuum. If (for measurement theory) the point in a continuum has no physical meaning, it is impossible to maintain that systems of this type behave in a deterministically predictable sense. The simplest example of such a system is the model (suggested by Einstein!) of a one-dimensional one-particle gas, where the mass-point is obliquely reflected from two straight boundaries ($x=0$, $x=l$). If its initial position and velocity x_0 and v_0 are uncertain within Δx_0, Δv_0, the uncertainty in its position Δx at time t equals $\Delta x=(\Delta v_0)t$ and after time $t_c=l/\Delta v_0$ the variation of x is larger than the whole range l of x.

If classical mechanics is formulated in this way, then some of the distinctions between the classical and quantum theories disappear. Thus, for example, the feature of

quantum mechanics that each measurement introduces new initial conditions (the so-called "reduction of probability") appears to be equally true for classical theory. The essential quantum effects that remain are the reciprocal relation between the maximum of sharpness for coordinate and velocity in the initial and subsequent states and the interference of probabilities. "For macrobodies both these effects remain small for a long time; during this time the individualistic description of traditional classical mechanics is a good approximation. But there is always a critical moment t_c when this ceases to be true, and the quasi-individual is transforming itself into a genuine statistical ensemble". *A. Salam.*

Bunge, Mario. Survey of the interpretations of quantum mechanics. Amer. J. Phys. **24** (1956), 272–286.

Popovici, Andrei. Le déterminisme quantique. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 105–147. (Romanian. Russian and French summaries)

Matsunobu, Hiroyuki; and Takebe, Hisao. Tables of U coefficients. Progr. Theoret. Phys. **14** (1955), 589–605.

The Kronecker (direct) product of three or more irreducible matrix representations of the 3-dimensional rotation group, R_3, can be reduced in more than one way into a direct sum of irreducible representations. In practical calculations in quantum mechanics it is frequently necessary to know the transformation matrices connecting different reduction schemes. A general algebraic theory has been developed by G. Racah [Phys. Rev. (2) **63** (1943), 367–382] but it is usually convenient to resort to tables. Let D_J ($J=0, \frac{1}{2}, 1, \cdots$) be the irreducible representations of R_3 and let $\times$ indicate the Kronecker product. Finally, let $D_{J'}\times D_{J''}\Rightarrow D_J$ represent the process of isolation of the irreducible component D_J in the reduction of the product to a direct matrix sum. The authors consider the two schemes

(a) $D_{\frac{1}{2}}\times D_l\Rightarrow D_j$, $D_{\frac{1}{2}}\times D_{l'}\Leftarrow D_{j'}$, $D_j\times D_{j'}\Rightarrow D_J$,
(b) $D_l\times D_{l'}\Rightarrow D_L$, $D_{\frac{1}{2}}\times D_{\frac{1}{2}}\Leftarrow D_S$, $D_L\times D_S\Rightarrow D_J$.

This gives the conversion from jj-coupling to LS-coupling for two particles of spin $\frac{1}{2}$, and vice versa. Tables of the elements of the transformation matrices are given for the special values $1/2\leqq(j, j')\leqq 7/2$ with L or j ranging from 0 to 4, inclusive. *E. L. Hill* (Minneapolis, Minn.).

Kurşunoğlu, Behram. Transformation of relativistic wave equations. Phys. Rev. (2) **101** (1956), 1419–1424.

According to the method of Foldy and Wouthuysen [Phys. Rev. (2) **78** (1950), 29–36] the Dirac equation for a free electron, $(\gamma_\varrho p_\varrho+\mu)\psi=0$, can be transformed by the unitary transformation $\varphi=e^{is}\psi$, $s=\frac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\mathbf{p}/\mu)$, into the form $i\hbar\partial\varphi/\partial t=c\beta(p^2+\mu^2)^{\frac{1}{2}}\varphi$, where $\gamma_\varrho\gamma_\sigma+\gamma_\sigma\gamma_\varrho=-2\delta_{\varrho\sigma}$ ($\varrho, \sigma=1, \cdots, 4$), $(\gamma_\varrho)=(\boldsymbol{\gamma}, i\beta)$, $(p_\varrho)=(-i\hbar\partial/\partial x_\varrho)=(\mathbf{p}, ip_0)$, $p=|\mathbf{p}|$, $\mu=mc$. In the usual representation, in which β is diagonal, the transformed equation is immediately split into positive- and negative-energy parts. It is shown that the Foldy-Wouthuysen transformation may be generalized in a fairly simple manner (without recourse to successive approximations) to include the cases in which the electron undergoes interactions with external fields as well as fluctuating quantized vacuum fields. The final equations for the positive- and negative-energy parts have respectively the forms

$$\text{(I)}\qquad \begin{aligned}(\beta M-R)\varphi_+ &+L(\beta M+R)^{-1}L\varphi_+=0,\\ (\beta M+R)\varphi_- &+L(\beta M-R)^{-1}L\varphi_-=0,\end{aligned}$$

where

$$\text{(II)}\qquad \begin{aligned}M&=2\mu\beta(U^2+U^{2\dagger})^{-1},\quad U=e^{-is},\\ R&=\tfrac{1}{2}(U^\dagger\pi_0U+U\pi_0U^\dagger),\\ L&=\tfrac{1}{2}(U^\dagger\pi_0U-U\pi_0U^\dagger).\end{aligned}$$

For electrodynamics

$$S=\tfrac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\pi}/\mu-i\mathfrak{M}_0/\mu),$$
$$(\pi_\varrho)=(\boldsymbol{\pi}, i\pi_0)=(p_\varrho-(e/c)A_\varrho^e),$$

where A_ϱ^e are the external electromagnetic potentials, while for neutral pseudoscalar meson theory

$$S=\tfrac{1}{2}\tan^{-1}(-i\boldsymbol{\gamma}\cdot\mathbf{p}/\mu+iG\gamma_5\varphi^e/\mu-i\mathfrak{M}_0/\mu),$$
$$\pi_0=p_0,$$

where φ^e is the external meson field. In both cases

$$\mathfrak{M}_0=\tfrac{1}{2}(\mathfrak{M}-\beta\mathfrak{M}\beta),$$

where $\mathfrak{M}$ is the "mass operator" for the system in question. Illustrative expansions of equations (I) and (II) are carried out for the two cases. An extension of the method to relativistic two-body equations is briefly sketched. *B. S. De Witt* (Chapel Hill, N.C.).

Koba, Z. Velocity of the Dirac electron. Nuovo Cimento (10) **3** (1956), 1–5.

Dirac's argument [Principles of quantum mechanics, 3rd ed., Oxford, 1947, p. 261; MR **9**, 319] concerning the velocity eigenvalues, $\pm c$, of a relativistic electron is analyzed. It is shown that an eigenfunction, with eigenvalue $+c$, of a given component of the velocity is a superposition of momentum eigenfunctions which has the special property that large positive values of momentum correspond solely to positive energy states while large negative values of momentum correspond solely to negative energy states. If the velocity eigenfunction is also taken to be a simultaneous eigenfunction of position, so that the spread of momenta is uniform from $-\infty$ to ∞ with large momentum values being consequently almost a certainty, this property, taken together with the fact that the velocity and momentum of a negative energy electron are oppositely directed, gives additional insight into the foundations of Dirac's argument. *B. S. De Witt.*

Koba, Z. Supplementary remark on my previous note "Velocity of the Dirac electron". Nuovo Cimento (10) **3** (1956), 214–215.

The paper reviewed above is amplified. It is pointed out that a relativistic electron wave function which, as a result of a physical measurement, is a δ-type eigenfunction of a position coordinate at the measurement time $t=0$, generally splits momentarily into two peaks traveling in opposite directions with velocities $\pm c$. A third peak, however, quickly arises at the original coordinate eigenvalue and smears out the two initial peaks by the time $t_c=\hbar/mc^2$. Only if a second position measurement is made before the time t_c has elapsed can the electron be "prepared" in a simultaneous eigenstate of position and velocity. This pair of measurements is just what Dirac describes as a velocity measurement and necessarily leads to the eigenvalues $\pm c$. *B. S. De Witt* (Chapel Hill, N.C.).

Jeffreys, Bertha. The use of the Airy functions in a potential barrier problem. Proc. Cambridge Philos. Soc. **52** (1956), 273–279.

The exact solution of Schrödinger's equation in one dimension for a single particle is given for a trapezoidal

potential well. The solution involves Airy functions (near the turning points) and trigonometrical functions. Numerical values are found for the ratio of the amplitudes of the transmitted and reflected waves. The results are compared with those derived using the WKB method.

N. D. Kazarinoff (Lafayette, Ind.).

Mac-Dowell, Samuel Wallace. Proca's equation in spherical coordinates. An. Acad. Brasil. Ci. **27** (1955), 279–284. (Portuguese)

The problem of separating the Proca equation for spin 1 particles in a central field is solved with the use of an operational method in spherical coordinates. A second-order differential radial equation for a two-components spinor, already discussed by Case [Phys. Rev. (2) **80** (1950), 797–806; MR **12**, 708] for a Coulomb potential, is obtained.

M. Cini (Turin).

Newton, Roger G. Remarks on scattering theory. Phys. Rev. (2) **101** (1956), 1588–1596.

It is pointed out that of the family of potentials $V(r)$ which are equivalent with respect to both phase shift and binding energies E_n a unique one can be selected by the requirement $\lim_{r\to\infty} V(r)e^{2Kr}=0$, $K^2=\max E_n$. An expression is derived for the variational derivative of the potential with respect to the lth phase shift, $\delta V(r)/\delta\eta_l$, and several results are obtained from it. Firstly, the equation $V(0)=8(\pi(2l+1))^{-1}\int_0^\infty \eta_l(k)k dk$ is shown to hold when there are no bound states and when both sides exist; the latter condition is in general not satisfied, since it implies $\int_0^\infty V(r)dr=0$. However, if V and V_0 have the same bound states and η_l and $\eta_l^{(0)}$ are asymptotically equal, then

$$V(0)=V_0(0)=8(\pi(2l+1))^{-1}\int_0^\infty[\eta_l(k)-\eta_l^{(0)}(k)]k\,dk$$

holds under correspondingly weaker conditions. Secondly, a "phase-shift perturbation method" can be developed from $\delta V/\delta\eta_l$ for both central and tensor forces. It corresponds to the Born approximation when there are no bound states. As "unperturbed" phase shift the author suggests the so-called "intrinsic" phase shift. If there is only one bound state this intrinsic phase shift is defined exactly by the shape-independent approximation of the effective range theory. The appendix contains a proof of the following theorem: A sufficient condition for a Bargmann potential (i.e. one for which the lth eigenvalue of the S-matrix is a rational function of k) to decay exponentially is that near $k=0$, $f_l(k)-f_l(0)=O(k^{2l})$.

F. Rohrlich (Iowa City, Ia.).

Klein, Abraham. Scattering matrix in the Heisenberg representation for a system with bound states. Progr. Theoret. Phys. **14** (1955), 580–588.

Every closed system of interacting fields will contain steady states Ψ_S, viz. the vacuum state, one-particle states, and possibly bound states. The central thought in this paper is the realization that for any scattering experiment the boundary condition in the remote past involves only a superposition of such states. Mathematically, this suggests the construction of a space X from the space of steady states only. This space X is the infinite direct product space formed from the Ψ_S. It is spanned by the eigenvectors Φ_n of an operator H_X which has the same spectrum as the total Hamiltonian H of the system. This important property implies that the Heisenberg eigenstates Ψ_n of H are related to the Φ_n by a unitary transformation U. For steady states $\Psi_S=\Phi_S$, Ψ_S and Φ_S being subsets of Ψ_n and Φ_n. Since X is assumed to be uniquely defined, the transformation U depends on the boundary conditions imposed on Ψ_n. In particular, $\Psi_n^{(\pm)}$ leads to $U^{(\pm)}$. The S-operator can then be defined, $S=U^{(-)}U^{(+)\dagger}$. The relation to the Yang-Feldman formalism is developed and the covariant amplitudes are used to express S-matrix elements.

F. Rohrlich (Iowa City, Ia.).

Segal, I. E. Uniqueness of the fine-structure constant. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 1103–1107.

This paper gives a very brief account of an extremely abstract and general approach to the quantum theory of fields, and it sketches an argument to the effect that in a covariant theory of interacting fields the coupling constant can have at most one single non-vanishing value. The approach is in terms of a general algebraic formulation of second quantization, as developped by J. M. Cook [Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825], and of the use of a fundamental group G embodying the Lorentz group and the transformations generated by the position operators in space-time. To each infinitesimal transformation X of G corresponds an operator $\Gamma(X)$ in the Hilbert space H of state vectors of the system of (two) interacting fields. One considers also the value $\Gamma_0(X)$ that $\Gamma(X)$ would have in absence of interaction, and the difference $I(X)=\Gamma(X)-\Gamma_0(X)$, called the interaction component of the operator $\Gamma(X)$, (we do not follow exactly the author's notation).

A fundamental postulate is then introduced, called postulate of dynamical covariance. It states that for any transformation a of G and any infinitesimal transformation X of this group, one has the two equivalent relations:

$$\Gamma(aXa^{-1})=A\Gamma(X)A^{-1},\ I(aXa^{-1})=AI(X)A^{-1},$$

$A=\Gamma_0(a)$ being that representation of a in the Hilbert space H which would result in absence of interaction. This postulate is justified on the basis of the assertion that in usual field theories each interaction term $I(X)$ is the same function of the free-field operators in all Lorentz frames [Note of the reviewer: This fact is actually valid for the densities of momentum, energy, angular momentum, etc. The quantities $\Gamma(X)$ and $I(X)$ however are in usual theories integrals of such densities on frame dependent hyperplanes of space-time. How this circumstance affects the postulate of dynamical covariance is not discussed by the author.] Once the above relations are accepted, the argument runs as follows. The relations imply the identity $iI([X, Y])=[I(X), I(Y)]$, where X, Y are two infinitesimal operators of G and $[\cdots]$ denotes the commutator. Since I is essentially linear in the coupling constant, this identity can hold at most for a single nonvanishing value of it.

According to a footnote, the present paper is largely a summary of a course given at the Department of Mathematics of the University of Chicago in 1955, notes on which have been duplicated. These notes are unfortunately as brief as the present paper on the centrally important points of dynamical covariance and uniqueness of coupling constant.

L. Van Hove (Utrecht).

Rzewuski, Jan. Differential structure of non-local theories. II. Acta. Phys. Polon. **14** (1955), 121–134. (Russian summary)

[For part I see same Acta **13** (1954), 135–144; MR **16**, 319.] In this paper the author makes an important advance in his study of non-local theories. It is shown that

every solution of the non-local equation

$$(\Box^2-m^2)q(x)=\lambda\int_{\sigma_1}^{\sigma_2}K(x,x')q(x')dx'$$

is also a solution of the equation

$$(\Box^2-m^2)q(x)=-\lambda\int_{\sigma}A([\sigma],x,x')(\partial_\mu^{+}-\partial_\mu^{-})q(x')d\sigma_\mu',$$

and vice versa, where $A([\sigma],x,x')$ are quantities defined on the surface σ in terms of the kernel $K(x,x')$. The essential point is that whereas the original integro-differential equation contains integrals over space-time, the equivalent equation containing the A's provide a description of the system which is local in time. Explicit formulae are given for an iterative construction of the A's and a number of special examples are discussed. *A. Salam.*

Percus, J. K. Inadmissible auxiliary conditions in quantized linear systems. Phys. Rev. (2) **100** (1955), 1208–1213.

Different methods of imposing auxiliary conditions on quantized linear systems are reviewed. It is shown that for systems obeying Fermi-Dirac statistics auxiliary conditions can always be made consistent with commutation relations by means of suitable modifications of the latter, while for systems obeying Bose-Einstein statistics such a procedure is possible only if positive definite quadratic constants of the motion exist. In this case a consistent formulation can be obtained by introducing a restricted set of new field variables in terms of which all pertinent field quantities may be expressed. *M. Cini.*

Imamura, Tsutomu. Potential in quantum field theory. Progr. Theoret. Phys. **13** (1955), 183–188.

The R-matrix is defined in terms of a phenomenological potential v by the equation $R=v+va^{-1}R$, where $a=E-H_0+i\epsilon$, [Lippmann and Schwinger, Phys. Rev. (2) **79** (1950), 469–480; MR **12**, 570]. For 2 fermion scattering the equation

$$(pp'|S-1|qq')=$$
$$-2\pi i\delta(H_0(p)+H_0(p')-H_0(q)-H_0(q'))(pp'|R|qq'),$$

gives the relation between the R and the S matrices. An integral equation in the manner of Tamm and Dancoff can be written for $(pp'|S-1|qq')$. The definitions above then allow for the construction of the potential v.

A. Salam (Cambridge, England).

Dirac, P. A. M. Gauge-invariant formulation of quantum electrodynamics. Canad. J. Phys. **33** (1955), 650–660.

A new form of quantum electrodynamics is developed, which is very general in the sense that the Lorentz condition on the potentials $\partial A^\mu/\partial x^\mu$ is not used, and that it is put in a form explicitly gauge invariant. This is achieved by introducing in the c-number theory a four-component gauge-invariant spinor ψ^* for describing the electrons, given by

$$\psi^*=\psi\exp(iC(x)),$$

where ψ is the usual spinor and $C(x)$ a suitably defined function of space-time coordinates, linear in the potentials. Quantization is obtained by the method already developed by the author in previous papers [e.g. Canad. J. Math. **2** (1950), 129–148; MR **13**, 306.] It is shown that the operator ψ^* is the operator of creation of an electron together with its Coulomb field, in contradistinction with the usual operator ψ, which creates the bare electron. Furthermore, a comparison is made between the new theory and some aspects of the usual one, showing partial equivalence under certain conditions. Finally some remarks are presented on the problem of charge quantization in connection with discrete Faraday lines of force, a notion linked to the gauge-invariant operator ψ^*.

D. Rivier (Lausanne).

Arnowitt, R.; and Deser, S. Renormalization of derivative coupling theories. Phys. Rev. (2) **100** (1955), 349–361.

The functional formulation of field theory [S. F. Edwards and R. E. Peierls, Proc. Roy. Soc. London. Ser. A. **224** (1954), 24–33; MR **15**, 1010] is used to investigate the problem of the renormalization of meson theories with derivative couplings. Contrary to the conclusions obtained by perturbation method, the theories with derivative couplings are found to be renormalizable, the divergences having the form of essential singularities and not the branch-point behavior, as in the theories renormalizable by perturbation method. The functional method shows also another important feature of the theory: the existence of a critical value for the coupling constant, above which analytic continuation is to be performed if the transformation theory must remain valid. However only a neutral meson field is considered and this restriction is essential for the method, which does not apply to charged mesons. Another simplification essential for the method is the omission of closed loops effect, especially of vacuum polarization. *D. Rivier* (Lausanne).

Cooper, Leon N. Some notes on nonrenormalizable field theory. Phys. Rev. (2) **100** (1955), 362–370.

The author presents an investigation of certain field theories, nonrenormalizable by the perturbation method, which uses the functional integral formulation first proposed by Feynman [Rev. Mod. Phys. **20** (1948), 367–387; MR **10**, 224] and exploited by Edwards and Peierls [Proc. Roy. London. Ser. Soc. A. **224** (1954), 24–33; MR **15**, 1010] and P. Matthews and A. Salam [Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 693]. Only theories with neutral meson field and with derivative coupling are investigated. Results are in agreement with those obtained by Arnowitt and Deser in the paper reviewed above in the sense that the theories investigated are found to be renormalizable, that is, it is possible to make all the operators of the theories finite by mass, Z_1 and Z_2 renormalizations. The reason why the perturbation method cannot renormalize these theories is the following: the (renormalized) propagator has at the origin of the co-ordinate space an essential singularity, and its Fourier transform exists only in one half of the complex coupling-constant plane and has a branch-point at the origin in that plane; it is therefore a many-valued function of the coupling constant. Infinities which occur in the theories investigated are found to be of two distinct classes: 1) those which have nothing to do with perturbation method, and which occur in a similar way in renormalizable and nonrenormalizable theories; 2) those whose existence is due to the singularities of the momentum-space propagator in the coupling-constant plane, characteristic of the theories nonrenormalizable by perturbation method. *D. Rivier* (Lausanne).

Caianiello, E. R. Perturbative expansions. Nuovo Cimento (10) **2** (1955), 155–159.

More results are obtained for fermion kernels previously

introduced by the author [Nuovo Cimento (9) **10** (1953), 1634–1652; **11** (1954), 492–529; MR **15**, 586; **17**, 440] which do not depend on perturbation theory and provide formal relations between the various kernels.

A. Salam (Cambridge, England).

Caianiello, E. R. Non-perturbative expansions. Nuovo Cimento (10) **2** (1955), 186–188.

The author has previously shown [Nuovo Cimento (9) **11** (1954), 492–529; **12** (1954), 561–562; (10) **1** (1955), 337–340; MR **17**, 440; **16**, 1185] (i) that all propagation kernels satisfy integro-differential relations which contain first and second derivatives of the kernel with respect to the interaction constant λ and (ii) that all λ-derivatives exist at $\lambda=0$. This paper shows that perturbative expansions for these kernels can be obtained under rather general conditions. *A. Salam* (Cambridge, England).

Edwards, S. F. The nucleon Green function in pseudoscalar meson theory. I, II. Proc. Roy. Soc. London. Ser. A. **232** (1955), 371–376, 377–389.

I. Matthews and Salam [Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 693] have shown that the functional integral

$$\int \exp\left(i\int \mathcal{L}(\bar{\psi}, \psi, \phi)d^4x\right)\delta\psi\delta\bar{\psi}\delta\phi$$

which generates the class of all Fermion-meson Green's functions, equals $\int\delta\phi\,\exp(-L(\phi)\exp i\int\phi\Delta^{-1}\phi d^4x$ after the Fermi functional integrations have been performed. Here $\exp -L(\phi)$ is essentially the Fredholm determinant $[\det(1+igS_F\gamma_5\phi)]$. The author shows that for the analogous case where ψ represents a charged Bose (rather than a Fermi) field, $\exp -L(\phi)$ equals the corresponding permanent (determinant with only positive signs before each combination of terms $a_{1\delta}a_{2\gamma}\cdots a_{n\alpha}$). The result that a change of statistics changes the character of $\exp -L(\phi)$ had already been recognised by Feynman [Phys. Rev. (2) **76** (1949) 749–759; also W. Pauli, Progr. Theoret. Phys. **5** (1950), 526–543; MR **12**, 572]. The remark about its connection with "permanents" is new.

II. This is the main paper, embodying the author's approach for performing the ϕ functional integration in part I. Basically the method is that of stationary phase, taken in its second approximation. It is claimed that the approximation is accurate both in the limits of strong and weak couplings. The pseudoscalarity of the meson, and its charge, introduce their peculiar difficulties. Each one of these needs a special approximation procedure which is justified by considering appropriate models. Combining these separate proposals, the evaluation of the functional integrals is expressed in terms of the solution of a set of coupled equations which provide a basis for a covariant intermediate coupling approach. The solution of these equations is deferred to a later paper. *A. Salam.*

Lehmann, H.; Symanzik K.; und Zimmermann, W. Zur Vertexfunktion in quantisierten Feldtheorien. Nuovo Cimento (10) **2** (1955), 425–432.

Consider the so called "vertex function" $\Gamma_5(p_1, p_2)$ in pseudoscalar meson theory. From general arguments of invariance it can be written in the form $\Gamma_5(p_1, p_2)=\gamma_5\cdot l(-(p_1-p_2)^2)$, if factors $i\gamma p_1$ to the left and $i\gamma p_2$ to the right can be put equal to $-M$ (M=nucleon mass). The authors show that if this theory has no anomalous states with negative probability of the kind encountered in the Lee model [T. D. Lee, Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317; see also the paper reviewed below] the function $f(\varkappa^2)$ defined above must fulfill the inequality

$$(1)\qquad \frac{1}{8\pi^2}\int_{4M^2}^{\infty}\frac{\varkappa(\varkappa^2-4M^2)^{\frac{1}{2}}}{(\varkappa^2-m^2)^2}|f(\varkappa^2)|^2 d(\varkappa^2)\leq 1$$

(m=meson mass).

The proof is based on the positive-definite character of a certain sum of absolute squares. The condition (1) is necessary but not sufficient to ensure the absence of the anomalous states. No attempt is made at a discussion of whether or not this condition is actually fulfilled.

G. Källén (Copenhagen).

Källén, G.; and Pauli, W. On the mathematical structure of T. D. Lee's model of a renormalizable field theory. Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7, 23 pp.

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V\rightleftarrows N+\theta$ but forbidding $N\rightleftarrows V+\theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/g_0^2=1-Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio g^2/g_0^2 should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR **13**, 713; Källén, Helv. Phys. Acta **25** (1952), 417–434; MR **14**, 435; Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332; Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value g_{crit}. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

The authors discover the remarkable result that if $g>g_{\text{crit}}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g>g_{\text{crit}}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

A. Salam (Cambridge, England).

Kahana, S.; and Coish, H. R. Classical meson theory. I. Amer. J. Phys. **24** (1956), 225–239.

Jouvet, B. A realistic theory of mesons. Nuovo Cimento (10) **2** (1955), supplemento, 941–968.

This paper is partial exposition of a broad undertaking to construct a theory in which all the "elementary" particles of physics are built out of a small number of

"primary" fermions of spin $\frac{1}{2}$ [following the ideas of L. de Broglie, Théorie générale des particules à spin (méthode de fusion) Gauthier-Villars, Paris, 1943]. The attempt is made to prove that a sum of biquadratic Fermi-type couplings between the primary fields, each with the same coupling constant, is equivalent to a Yukawa-type theory in which the interactions between the fermions take place via bosons of integral spin, the renormalized boson propagators here being replaced, at least approximately, by interaction kernels arising from virtual fermion pairs. The results obtained depend crucially on a certain method of summing the S-matrix perturbation series and on the use of regulators [W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949), 434–444; MR **11**, 301] the latter being necessary owing to the "nonrenormalizability" of Fermi-type theories. Thus the renormalized boson-fermion coupling constants for the equivalent Yukawa theory here depend on largely arbitrary regularization constants, much of the physical content of the theory being therefore introduced not at the outset but via the technical details of a perturbation expansion. A note added in proof contains (1) corrections and additions to the text; (2) a brief comparison of the theory with some recent ideas of Heisenberg; (3) some conjectures on selection rules.

B. S. De Witt (Chapel Hill, N.C.).

Arnous, E. Le problème des états liés et le modèle d'interaction nucléaire par paires de mésons (modèle de Wentzel). J. Phys. Radium (8) **17** (1956), 107–111.

The question of the existence of bound states in the meson-pair theory of Wentzel [Helv. Phys. Acta **15** (1942), 111–126] is examined. It is suggested that their appearance depends on an ambiguity in the choice of coupling-constant renormalization. With one such choice a bound physical state is found; with another, only pathological states. It is claimed that in the limit of a point nucleon source the first choice behaves properly. The author mentions that bound states occur only for a negative coupling constant, but the fact that the Hamiltonian is then no longer positive definite is disregarded. The author seems unaware also of the proof by Klein and McCormick [Phys. Rev (2) **98** (1955), 1428–1445; MR **17**, 442] that even with his first choice of renormalization, the theory becomes pathological in the point-source limit.

S. Deser (Cambridge, Mass.).

Scarf, Frederick L. Spectrum and nonrelativistic limit of a Bethe-Salpeter equation. Phys. Rev. (2) **100** (1955), 912–923.

Geffen, D. A.; and Scarf, F. L. Spurious solutions of a Bethe-Salpeter equation. Phys. Rev. (2) **101** (1956), 1829–1830.

The first paper studies the exact solution to a Bethe-Salpeter equation between two scalar "nucleons" interacting through a scalar, massless field. Especially, he argues that the coupling constant, which is considered to be the eigenvalue parameter, tends to zero with the binding energy for every solution in the discrete spectrum. This is in disagreement with previous investigations [G. C. Wick, Phys. Rev. (2) **96** (1954), 1124–1134; R. E. Cutcosky, Phys. Rev. (2) **96** (1954), 1135–1141], where it was found that some of the solutions had finite limits of the coupling constant for zero binding energy. They were therefore rejected as being unphysical. The reason for the discrepancy in the results is discussed and found to be due to different treatments of the very intricate limit of zero binding energy by the different authors. The properties of the wave functions are discussed in some detail, and it is argued that there is no reason to reject any of the solutions as being unphysical. However, in the second article, by Geffen and Scarf [see also the paper reviewed below], the authors point out a mistake in the original calculation. When this mistake is corrected, the results of all investigations agree, and the Bethe-Salpeter equation still appears to have unphysical solutions of the kind found by Wick and Cutcosky.

G. Källén (Copenhagen).

Cutkosky, R. E.; and Wick, G. C. Spectrum of a Bethe-Salpeter equation. Phys. Rev. (2) **101** (1956), 1830–1831.

In the two previous papers by G. C. Wick and R. E. Cutcosky cited in the preceding review the authors have studied the exact solutions of a simplified equation of the Bethe-Salpeter type and found that some of the mathematical solutions are unphysical in the sense that the coupling constant does not tend to zero with the binding energy. As some doubts have recently been raised in this connection [see the paper reviewed above], the authors here give an independent argument with the result that all solutions except two (for a given value of one of the quantum numbers in the problem) must have the coupling constant larger than the finite value 1/4 for all binding energies. The new proof is based on a comparison between the eigenvalue equation of the problem and a simple hypergeometric equation that can be solved explicitly and the solutions of which must have the same number of zeros as the given equation. *G. Källén.*

Feldman, David. Nonrelativistic interaction between two nucleons. Phys. Rev. (2) **98** (1955), 1456–1470.

This is the definitive paper for the solution of the problem of deriving a two-nucleon Schrödinger equation from the conventional Tamm-Dancoff formalism. In the simplest Tamm-Dancoff approximation, there are two coupled equations involving the 2-nucleon (Ψ_0) and 2-nucleon and 1-meson amplitudes (Ψ_1). The amplitude Ψ_1 is eliminated and the resulting linear integral equation for Ψ_0 is identified as a genuine Schrödinger equation for the 2-nucleon system. This identification is objectionable, because the interaction operator usually depends on the energy of the system, so that the solutions of this equation are not orthogonal to one another. The commonly used iteration procedure [M. M. Lévy, Phys. Rev. (2) **88** (1952), 72; MR **14**, 706; A. Klein, ibid. **90** (1953), 1101–1115; MR **15**, 844] to remove the explicit energy dependence only succeeds in making this operator non-hermitian.

Remarking that the present problem of reconciling a relativistic field-theoretic description with an essentially non-relativistic and phenomological one, is closely parallelled in another problem in quantum mechanics, viz. the problem of reducing the 4-component Dirac equation for a particle in an external field to a 2-component Pauli equation, the author suggests the use of methods similar to the ones developed for the latter problem [L. L. Foldy and S. A. Wouthuysen, ibid. **78** (1950), 29–36]. The two-particle Tamm-Dancoff amplitude is decoupled from all the others by a series of canonical transformations, thus avoiding all difficulties of energy-dependent and non-Hermitian potentials. It is stressed that the success of this approach depends on assuming that the coupling to high-momentum states of virtual mesons is weak. The results of Lévy and Klein are reproduced, the range of their applicability being clearly defined. *A. Salam* (Cambridge, England).

Takabayasi, T. Variational principle in the hydrodynamical formulation of the Dirac field. Phys. Rev. (2) **102** (1956), 297–298.

Tiomno, J. Relativistic theory of spinning point particles. An. Acad. Brasil. Ci. **27** (1955), 259–269.

The non-relativistic classical theory of spinning particles is reformulated in terms of two-component spinors. A relativistic generalization of the theory can be obtained by making a variation integral Lorentz-invariant and by introducing four-components spinors. Two alternatives are possible, corresponding to Hamiltonians quadratic or linear in the momenta. The latter case is discussed in this paper and is shown to lead essentially to the same theory as that developed by Proca [J. Phys. Radium (8) **15** (1954), 65–72; MR **15**, 836]. The only difference consists in two supplementary conditions which restrict somehow the allowed solutions of Proca's equation. The transition to quantum theory gives, for spin $\frac{1}{2}$ and 1, the Dirac and the Kemmer-Proca-Yukawa equations respectively.

M. Cini (Turin).

Rayski, J. On the meaning of bilocalizability. Nuovo Cimento (10) **2** (1955), 255–272.

Consider the "quantized" space-time manifold $\mathbf{x}_\mu$, where $\mathbf{x}_\mu$ are operators. The corresponding infinitesimal displacement operators $il^2\mathbf{d}_\mu$ satisfy the commutation relation $[\mathbf{x}_\mu, il^2\mathbf{d}_\nu]=il^2\delta_{\mu\nu}$, where l is a fundamental length. The operators $-i\hbar\mathbf{d}$ are identified with the momentum $\mathbf{P}$ while the equality between $-i\hbar d_4$ and p_4 is regarded as a condition upon the state vector, thus resolving the difficulty that between energy and momentum there exists a physical relation while the $\mathbf{d}_\mu$'s are independent. The field variables ψ are operators constructed of the elements of quantised space-time, $\psi=\psi(\mathbf{x}_\mu, il^2\mathbf{d}_\mu)$. The connection with Yukawa's bipoint model [Phys. Rev. (2) **77** (1950), 219–226; MR **11**, 567] is made in the x-representation, where $\psi=(x'|\psi|x'')$. The field operators, then, are non-local; the local case arises when $l\to 0$. Or in Yukawa's notation, the traditional point model of elementary particles is obtained from the bipoint model by the supplementary conditions $r_\mu\Psi=0$, where $r_\mu=\frac{1}{2}(x_\mu'-x_\mu'')$.

The author considers this supplementary condition too restrictive. It is proposed to replace it by the conditions $(p_\mu r_\mu)\Psi=0$ and $[(al)^4p_\mu^2+r_\mu^2]\Psi=0$. The theory now postulates for every elementary particle a fundamental relation of the form $p_\mu^2+4k_\mu^2=0$ to replace the conventional Einstein relation. Here $p_\mu=\frac{1}{2}(p'+p'')_\mu$, $k_\mu=\frac{1}{2}(p'-p'')_\mu$, while k_μ^2 plays the role of mass. This relation combined with the supplementary conditions above leads to a generalization of the Schrödinger-Gordon equation $(p_\mu^2+4(k_\mu^2+r^{-2}[(al)^2(p_\mu k_\mu)^2-r_\mu r_\nu k_\mu k_\nu+2ir_\mu k_\mu)])\Psi(x, r)=0$ for the "rotator" model of an elementary particle. Among other results it is found that the square of the internal spin motion determines the rest masses of the particles, $M_n=(al)^{-1}[4n(n+1)]^{\frac{1}{2}}$ [see the following review].

A. Salam (Cambridge, England).

Rayski, Jerzy. On a bilocal theory of families of elementary particles. Acta Phys. Polon. **14** (1955), 107–120. (Russian summary)

The generalized bilocal Schrödinger-Gordon equation [see the preceding review] restricted by reciprocally invariant [Born, Rev. Mod. Phys. **21** (1949), 463–473, p. 465] supplementary conditions is solved, leading to a mass spectrum for the elementary particles. A close connection with Fierz's theory of higher-spin particles is established [Helv. Phys. Acta **12** (1939), 3–37; **23** (1950), 412–416; MR **12**, 67].

The author withdraws his earlier suggestion [Acta Phys. Polon. **13** (1954), 77–88; Nuovo Cimento (9) **10** (1953), 1729–1735; MR **15**, 766] that the rotations of the internal variables r_μ in his theory correspond to the rotations in isotopic space [A. Pais, Physica **19** (1953), 869–887; MR **15**, 766]. The variables r_μ are connected with ordinary space, as evidenced by the postulates of reciprocal invariance, while current hypotheses do not assume any connection between ordinary and isotopic-spin spaces.

A. Salam (Cambridge, England).

Potier, Robert. Sur les champs quantifiés de particules à spin quelconque. Relations de commutation. C. R. Acad. Sci. Paris **242** (1956), 1694–1697.

Halbwachs, Francis. Modèle de fluide relativiste décrivant les ondes du corpuscule de spin 0. C. R. Acad. Sci. Paris **242** (1956), 1689–1691.

Costa de Beauregard, Olivier. Particule libre à spin: complément à notre théorie covariante relativiste du produit scalaire hermitien. C. R. Acad. Sci. Paris **242** (1956), 1581–1583.

Costa de Beauregard, Olivier. Particule libre à spin: utilisation du projecteur d'Umezawa et Visconti dans notre formalisme covariant d'intégrales triples. C. R. Acad. Sci. Paris **242** (1956), 1692–1694.

Lepore, Joseph V. Isotopic spin of antiparticles. Phys. Rev. (2) **101** (1956), 1206–1207.

The eigenvalues of T_3, the diagonal component of the nucleon isotopic spin, are usually taken as $+1$, -1, for proton and neutron respectively. The assignment for antinucleons is found by imposing the conditions that a theory involving nucleons, meson and the electromagnetic field be invariant under electric charge conjugation. In this case the eigenvalues corresponding to antiprotons and antineutrons must be $+1$ and -1 respectively. However it should be pointed out that the 3-component of the total isotopic spin T_3 for a system containing nucleons and antinucleons is, in such a theory, given by:

$$T_3=T_3(\text{particles})-T_3(\text{antiparticles}).$$

M. Cini (Turin).

Itô, Hiroshi. Remarks on quantum hydrodynamics. Progr. Theoret. Phys. **13** (1955), 543–554.

This paper is concerned with Landau's theory of liquids, especially with the derivation of Landau's formulae for the energy spectrum of the rotons. It starts with critical remarks on one of Ziman's papers [Proc. Roy. Soc. London. Ser. A. **219** (1953), 257–269], concerning first his cut-off method for eliminating an infinity in the energy of the roton, and secondly his pick-up of only one term of the fourth order in the Hamiltonian. A quantum theory of incompressible liquid is developed, which emphasizes the importance of gauge invariance. Applied to the case of one roton present, it gives $E(p)=(2/3m)p^2$ for the roton energy, instead of $E(p)=\Delta+(3/8m)p^2$ as found by Ziman. The constant term Δ in Ziman's expression is shown to be due to the longitudinal motion and not to the lack of atomicity of his theory.

D. Rivier (Lausanne).

Reulos, René. Recherches sur la théorie des corpuscules. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 455–568 (1 plate) (1955).

Essentially a review of classical and quantum theories of the electro-magnetic field, leading on to equations involving matter also (Proca, Gordon etc.). One pleasant feature is that every equation is written out in full, no concession being made to summation conventions. It is sobering to look at some of the equations. *A. Salam.*

Ruedenberg, Klaus; Roothaan, C. C. J.; and Jaunzemis, Walter. Study of two-center integrals useful in calculations on molecular structure. III. A unified treatment of the hybrid, Coulomb, and one-electron integrals. J. Chem. Phys. **24** (1956), 201–220.

Continuing earlier investigations [same J. **19** (1951), 1445–1458, 1459–1477; MR **13**, 843] the authors give a general treatment of hybrid integrals

$$\int\int \chi_a'(1)\chi_a''(1)(r_{12})^{-1}\chi_a(2)\chi_b(2)\,dv_1dv_2,$$

including tables to facilitate the reduction of such integrals to standard integrals, and a bibliography of hybrid integrals. *A. Erdélyi* (Pasadena, Calif.).

See also: Günther, p. 815; Bognár and Freud, p. 852; Urbanik, p. 867; Łopuszański, p. 867; Gröbner, p. 911.

Thermodynamics, Statistical Mechanics

Tolmačev, V. V. Distribution functions with time correlations in the statistical mechanics of classical systems. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 439–441. (Russian)

This note is devoted to the definition in classical statistical mechanics of multiple distribution functions depending on several time arguments. The function of order s describes the joint probability of finding some particle in one element of phase space at one time, some particle (different or not from the first) in another element at another time, and so on to the sth particle. This definition is an extension to several time arguments of the familiar multiple distribution functions defined by Bogolyubov [Problems of dynamical theory in statistical physics, Gostehizdat, Moscow-Leningrad, 1946; MR **13**, 196] and by Born and Green [Proc. Roy Soc. London. Ser. A. **188** (1946), 10–18; MR **9**, 402], as well as a generalization to higher orders (for classical systems) of the pair correlation in space and time defined by the reviewer [Phys. Rev. (2) **95** (1954), 249–262, 1374–1384; MR **16**, 206]. A generating function is given for the distribution functions. It is used to derive a hierarchy of equations connecting the function of order s with s different time arguments to the $(s+1)$-th order function with two of its $s+1$ times put equal. *L. Van Hove* (Utrecht).

Lurçat, François. Sur la définition, en mécanique statistique, de l'entropie des états de non-équilibre. C. R. Acad. Sci. Paris **242** (1956), 1686–1688.

Maravall Casesnoves, Darío. Theoretical investigations of the random functions of microphysics. Rev. Mat. Hisp.-Amer. (4) **14** (1954), 118–137. (Spanish)

By an (incorrect) application of probability theory Maxwell's velocity distribution is "rediscovered", but with an average kinetic energy that increases linearly with time. Fortunately the author asserts that this effect is so small that thermodynamics is still useful. A few other applications to physics are discussed. *N. G. van Kampen.*

Born, M.; und Hooton, D. J. Statistische Dynamik mehrfach periodischer Systeme. Z. Physik **142** (1955), 201–218.

The authors study the statistical aspect introduced in classical mechanics by the assumption that the initial conditions are given with a non-vanishing uncertainty. This problem had been considered earlier by Born for a particle moving in one dimension between two reflecting walls [Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955) no. 2; MR **17**, 923]. The treatment given here is valid for all multiply periodic systems and makes use of action and angular variables, in terms of which the time evolution of the system is of course especially simple to express. One assumes the statistical distribution of initial conditions to be gaussian in these variables and determines its variation with time. The distribution remains gaussian in some sense (when the periodicity in the angular variables is disregarded) and the product of the widths when properly defined remains constant. If one considers the distribution in the angular variables alone it is found to become uniform after a sufficiently long time. *L. Van Hove.*

Bellman, Richard. Dynamics of a disordered linear chain. Phys. Rev. (2) **101** (1956), 19.

Dyson has determined the distribution function of characteristic frequencies of a linear chain of particles coupled harmonically, for a random distribution of masses and force constants [Phys. Rev. (2) **92** (1953), 1331–1338; MR **15**, 492]. The present paper gives a simplified derivation of the main mathematical formula used by Dyson. *L. Van Hove* (Utrecht).

Hijmans, J.; and De Boer, J. An approximation method for order-disorder problems. I, II, III. Physica **21** (1955), 471–484, 485–498, 499–516.

A procedure for order-disorder problems is given, which the authors describe as a systematic development of the methods of Kikuchi.

In a given lattice, a basic figure is chosen and analyzed relative to possible states of occupancy and possible subfigures. The energy for any given occupancy possibility for the lattice can be approximately described by such an analysis. The free energy is obtained by maximizing this expression for the energy relative to certain frequency parameters. These parameters are subject to restraints and by means of a change of variables, the description of the situation is made dependent on parameters describing the frequency of certain fully occupied figures. Other combinations are also introduced in order to obtain the simplest formulations for the maximum problem. For a triangular lattice the procedure is applied to the case in which the fundamental figure is a triangle and to the case of a rhombus, i.e. two triangles with a common side.

As Kikuchi has already pointed out the rhombus case does not contain more information than the triangle case. Figures such as the rhombus which have this property are termed "open". The authors point out that they must not be disregarded as subfigures. A detailed comparison is made with the Bethe method, the procedures of Guggenheim and McGlashan, and those of Kikuchi and Fosdick and James. *F. J. Murray* (New York, N.Y.).

See also: Mathews and Srinivasan, p. 865.